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On the Diagonals of Projections in Matrix Algebras Over Von Neumann Algebras

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Abstract
The main focus of this dissertation is on exploring methods to characterize the diagonals of projections in matrix algebras over von Neumann algebras. This may be viewed as a non-commutative version of the generalized Pythagorean theorem and its converse (Carpenter's Theorem) studied by R. Kadison. A combinatorial lemma, which characterizes the permutation polytope of a vector in $\mathbb{R}^n$ in terms of majorization, plays an important role in a proof of the Schur-Horn theorem. The Pythagorean theorem and its converse follow from this as a special case. In the quest for finding a non-commutative version of the lemma alluded to above, the notion of C*-convexity looks promising as the correct generalization for convexity. We make generalizations and improvements of some results known about C*-convex sets.

We prove the Douglas lemma for von Neumann algebras and use it to prove some new results on the one-sided ideals of von Neumann algebras. As a useful technical tool, a non-commutative version of the Gram-Schmidt process is proved for finite von Neumann algebras. A complete characterization of the diagonals of projections in full matrix algebras over an abelian C*-algebra is provided in chapter 5. In chapter 6, we study the problem in the case of M_2(M_n(C)), the full algebra of 2 x 2 matrices over M_n(C). The example gives us hints regarding the possibility of extracting an underlying notion of convexity for C*-polytopes, which are not necessarily convex.

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Dedicated to my parents, Soudamini Nayak and Sachidananda Nayak.

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ABSTRACT

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The main focus of this dissertation is on exploring methods to characterize the diagonals of projections in matrix algebras over von Neumann algebras. This may be viewed as a non-commutative version of the generalized Pythagorean theorem and its converse (Carpenter's Theorem) studied by R. Kadison. A combinatorial lemma, which characterizes the permutation polytope of a vector in $\mathbb{R}^n$ in terms of majorization, plays an important role in a proof of the Schur-Horn theorem. The Pythagorean theorem and its converse follow from this as a special case. In the quest for finding a non-commutative version of the lemma alluded to above, the notion of C^*-convexity looks promising as the correct generalization for convexity. We make generalizations and improvements of some results known about C^*-convex sets.

We prove the Douglas lemma for von Neumann algebras and use it to prove some new results on the one-sided ideals of von Neumann algebras. As a useful technical tool, a non-commutative version of the Gram-Schmidt process is proved for finite von Neumann algebras. A complete characterization of the diagonals of projections in full matrix algebras over an abelian C^*-algebra is provided in chapter 5. In chapter

6, we study the problem in the case of $M_2(M_n(\mathbb{C}))$, the full algebra of 2×2 matrices over $M_n(\mathbb{C})$. The example gives us hints regarding the possibility of extracting an underlying notion of convexity for C^*-polytopes, which are not necessarily convex.

Contents

Chapter 1

Introduction

1.1 The Pythagorean theorem and Carpenter's Theorem

The Pythagorean theorem in the Euclidean plane, in essence, says that if e_1, e_2 are the standard basis vectors for the Euclidean plane $\mathbb{R}^2$, then for a unit vector $x \in \mathbb{R}^2$, we have $|\langle x, e_1\rangle|^2 + |\langle x, e_2\rangle|^2 = 1$. One may restate the theorem as a criterion for possible lengths of projections $(\langle e_i, x\rangle, i = 1, 2)$ of the standard orthonormal basis vectors onto a one-dimensional subspace of $\mathbb{R}^2$ (spanned by x). Along with its converse (Carpenter's Theorem), it gives necessary and sufficient conditions for the lengths of those projections. Let E be the matrix associated with this projection (and the standard basis). Then the diagonal elements of E, given by $\langle e_i, Ee_i\rangle = \langle Ee_i, Ee_i\rangle = \|Ee_i\|^2, i = 1, 2$, are the squares of lengths of the projections. Thus these classical theorems may be generalized by posing questions about lengths of projections of standard basis vectors onto subspaces of $\mathbb{C}^n$, or equivalently diagonals of projections in $M_n(\mathbb{C})$. Below, we state Theorem 6 in [10] as an illustration for such a result.

Theorem 1.1.1. *Let φ be the mapping that assigns to each self-adjoint $n \times n$ matrix (a_{jk}) the point $(a_{11}, \cdots, a_{nn})(= \tilde{a})$ in $\mathbb{R}^n$, $\mathscr{K}_m$ be the range of φ restricted to the set $\mathscr{P}_m$ of projections of rank m, where $m \in \{0, \cdots, n\}$, and $\mathscr{K}$ be the range of φ restricted to the set $\mathscr{P}$ of projections. Then $\tilde{a} \in \mathscr{K}_m$ if and only if $0 \leq a_{jj} \leq 1$, for each j and $\sum_{j=1}^{n} a_{jj} = m$, and $\tilde{a} \in \mathscr{K}$ if and only if $0 \leq a_{jj} \leq 1$, for each j, and $\sum_{j=1}^{n} a_{jj} \in \{0, \cdots, n\}$.*

The version of the Pythagorean theorem and its converse, for the set of $n \times n$ complex matrices $M_n(\mathbb{C})$ mentioned above, may be thought of as a special instance of the Schur-Horn theorem. This classical result describes the set of diagonals of Hermitian matrices with a prescribed set of eigenvalues counted with multiplicity. The Pythagorean theorem corresponds to the case where the list of eigenvalues is given by the components of the vector $(1, \cdots, 1, 0, \cdots, 0)$ in $\mathbb{R}^n$, the number of 1's being the rank of the projection. It is worthwhile to note here that the set of diagonals of rank m projections in $M_n(\mathbb{C})$ is a convex set.

In [11], R. Kadison characterizes the diagonals of projections in $\mathcal{B}(\mathscr{H})$, the set of bounded operators on an infinite-dimensional complex Hilbert space. We quote the three relevant theorems, Theorem $13, 14, 15$ from [11] below, which characterize the diagonals of projections with finite-dimensional range, cofinite-dimensional range, and those with infinite-dimensional range whose complement is also infinite-dimensional, respectively.

Theorem 1.1.2. *If $\{e_b\}_{b\in\mathbb{B}}$ is an orthonormal basis for the Hilbert space $\mathscr{H}$ and numbers t_b in $[0,1]$ are specified, there is an m-dimensional subspace $\mathscr{V}$ of $\mathscr{H}$ such that $\|Fe_b\|^2 = t_b$ for each b in $\mathbb{B}$, where F is the projection with range $\mathscr{V}$, if and only if $\sum_{t_b\in\mathbb{B}} t_b = m$.*

Theorem 1.1.3. *If $\{e_a\}_{a\in\mathbb{A}}$ is an orthonormal basis for the Hilbert space $\mathscr{H}$ and $\{t_a\}_{a\in\mathbb{A}}$ is a family of numbers in $[0,1]$, there is an infinite-dimensional subspace $\mathscr{V}$ of $\mathscr{H}$ with m-dimensional orthogonal complement such that $\|Fe_a\|^2 = t_a$ for each a in $\mathbb{A}$, where F is the projection with range $\mathscr{V}$, if and only if $\sum_{a\in\mathbb{A}} 1 - t_a = m$.*

Theorem 1.1.4. *Let $\{e_1, e_2, \cdots\}$ be an orthonormal basis for the Hilbert space $\mathscr{H}$ and numbers $a_1, a_2, \cdots$ in $[0,1]$ be specified. Let $a'_1, a'_2, \cdots$ be the a_j in $(\frac{1}{2}, 1]$, $a''_1, a''_2, \cdots$ those in $[0, \frac{1}{2}]$, a the sum of the a''_j and b the sum $\sum_{j=1}^{\infty} 1 - a'_j$. There is an infinite-dimensional subspace $\mathscr{V}$ of $\mathscr{H}$ with infinite-dimensional complement such that $\|Fe_j\| = a_j$ for each j, where F is the projection with range $\mathscr{V}$, if and only if $\sum_{j=1}^{\infty} a_j$ and $\sum_{j=1}^{\infty} 1 - a_j$ diverge and either of a or b is infinite of both are finite and $a - b$ is an integer.*

The map $\varphi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$, which takes a matrix A to a diagonal matrix D with the same diagonal entries as that of A, is a trace-preserving conditional expectation from $M_n(\mathbb{C})$ to a maximal abelian self-adjoint algebra, namely the algebra of diagonal matrices. If $\mathscr{M}$ is a II_1 factor with a m.a.s.a. $\mathscr{A}$, then this point of view can be directly carried forward. Let $\Phi : \mathscr{M} \to \mathscr{A}$ be the unique trace-preserving normal conditional expectation. Then every positive contraction A (i.e. $0 \leq A \leq I$) in $\mathscr{A}$ ought to be the image of a projection E in $\mathscr{M}$ under Φ. However, this remains an open question.

Continuing in a similar vein, we are interested in characterizing the diagonals of projections in $M_n(\mathscr{R})$ where $\mathscr{R}$ is a von Neumann algebra. This may be viewed as a non-commutative version of the Pythagorean theorem as the entries of the matrices are from von Neumann algebras which are not necessarily $\mathbb{C}$.

1.2 Overview

In the next chapter, we describe the notation and conventions used in this document. We also set up the necessary background and context for the results in the later chapters, discussing the notions of numerical ranges, convexity and C^*-convexity. We include a proof of the Schur-Horn theorem and also describe certain generalizations. A combinatorial lemma, Lemma 5 in [10], is crucial to the proof. In the general case, we mimic the proof and study the impediments that are brought forth by the non-commutativity of the von Neumann algebra from which the entries of the self-adjoint matrix are derived.

In Chapter 3, we state and prove three technical lemmas that will be pivotal to our main results. The first lemma is an inequality, based on the Cauchy-Schwarz inequality, which gives upper bounds for the off-diagonal entries of an orthogonal projection in $M_n(\mathbb{C})$, which depend on the diagonal entries. Next we prove the Douglas lemma in the case of von Neumann algebras which relates the notions of majorization, and factorization of operators, in a von Neumann algebra. The third lemma may be viewed as a non-commutative Gram-Schmidt process for finite von Neumann algebras.

In Chapter 4, we discuss applications of the Douglas lemma to the theory of one-sided ideals of von Neumann algebras. We also apply it to generalize a result on C^*-convexity by Loebl-Paulsen, Theorem 15 in [14], to the setting of finite von Neumann algebras.

In Chapter 5, we consider the problem of characterizing diagonals of projections in $M_n(\mathfrak{A})$, where $\mathfrak{A}$ is an C^*-algebra. A result by Grove-Pedersen in [3] discusses topological obstructions for diagonalizing matrices over $C(X), X$ being a compact, Hausdorff space. In view of this, one may be tempted to believe that there ought to be similar obstructions to the problem of characterizing diagonals of projections in $M_n(C(X))$. In this chapter, we construct a projection with a prescribed set of diagonal entries, as mentioned in Theorem 7 in [10]. We prove that the construction is a continuous function on $\mathscr{K}_m$.

In Chapter 6, we take a dive into the non-commutative world. In the first, we work out the details in the case of $M_2(M_n(\mathbb{C}))$. The problem becomes one of characterizing the principal $n \times n$ diagonal blocks of projections in $M_{2n}(\mathbb{C})$. In remark 6.2.6, we note that there is an undercurrent of convexity, which is not as straightforward as in the commutative case. In the third section, we prove that any diagonal element of a projection in $M_n(\mathscr{R})$, where $\mathscr{R}$ is a countably decomposable von Neumann algebra, must be a C^*-convex combination of projections. For a finite factor $\mathscr{M}$, the diagonal elements of a trace a projection must be C^*-convex combinations of $I, 0$ and P for a projection P in $\mathscr{M}$ with trace $\frac{\{na\}}{n}$. As a word of caution, these results do not completely characterize the full diagonals of projections but rather the set of possible diagonal entries.

Chapter 2

Background and Context

2.1 Conventions and Terminology

We shall denote a complex Hilbert space by $\mathscr{H}$ and the set of bounded operators on $\mathscr{H}$ by $\mathcal{B}(\mathscr{H})$. Unless otherwise stated, for us, a Hilbert space will be over the field of complex numbers. The family $\mathcal{B}(\mathscr{H})$ is an algebra relative to the usual addition and multiplication (composition) of operators. With $\| \cdot \|$ the *bound* of B, $\mathcal{B}(\mathscr{H})$ provided with this norm becomes a Banach algebra. A family Γ of operators on $\mathscr{H}$ is said to be"self-adjoint" when A^*, the adjoint-operator of A, is in Γ if A is in Γ. The norm-closed subalgebras of $\mathcal{B}(\mathscr{H})$ are called "C*-algebras" and those closed in the strong-operator topology on $\mathcal{B}(\mathscr{H})$ (the topology corresponding to convergence of nets of bounded operators on individual vectors in $\mathscr{H}$) are the "von Neumann algebras" Our von Neumann algebras are required to contain the identity operator I

on $\mathscr{H}$ (so, $Ix = x$, for each x in $\mathscr{H}$.) We often denote a von Neumann algebra by "$\mathscr{R}$" and a C*-algebra by "$\mathfrak{A}$."

We use the following standard notation:

$\mathbb{N} \to$ the set of natural numbers,

$\mathbb{R} \to$ the set of real numbers,

$\mathbb{C} \to$ the set of complex numbers,

$M_n(\mathbb{C}) \to$ the set of $n \times n$ complex matrices.

We often write $\mathscr{R} \otimes M_n(\mathbb{C})$ as $M_n(\mathscr{R})$ and think of it as the set of $n \times n$ matrices with entries from the von Neumann algebra $\mathscr{R}$.

2.2 Numerical Ranges and Convexity

The numerical range $W(T)$ of a bounded operator T on a complex Hilbert space $\mathscr{H}$, is the set of complex numbers of the form $\langle x, Ax \rangle$, where $x \in \mathscr{H}$ is a unit-norm vector.

$$W(T) := \left\{ \frac{\langle x, Tx \rangle}{\langle x, x \rangle} \ : \ x \in \mathscr{H} - \{0\} \right\}.$$

This subset of the complex plane $\mathbb{C}$ may be thought of as a *photograph* of T, as it succinctly captures information about the eigenvalues, algebraic, analytic structure of T in the geometry of its boundary. An illustration of such a result is the following theorem which we state without proof.

Theorem 2.2.1. *Suppose that the numerical range of $A \in M_n(\mathbb{C})$ is a convex* *polygon. Then the following hold :*

(i) If $n \leq 4$, then A is normal.

(ii) If $n > 4$, then either A is normal, or A is unitarily similar to a matrix of the form $B \oplus C$, where B is normal and $W(C) \subseteq W(B)$.

(iii) If $W(A)$ has n or $n-1$ vertices, then A is normal.

The boundary of the numerical range contains all of its essential information because of the Toeplitz-Hausdorff theorem which states that the numerical range of a bounded operator is a convex set. In the original proof (for $A \in M_n(\mathbb{C})$), in [18], Toeplitz first proved that the outer boundary is a convex curve and in [6] Hausdorff proved that $W(A)$ is simply connected. For the sake of completion, we include a well-known proof below which is different from the original proof.

Theorem 2.2.2 (Toeplitz-Hausdorff)**.** *For any operator T in $\mathcal{B}(\mathscr{H})$, $W(T)$ is convex.*

Proof. First we prove this in the case where $\mathscr{H} = \mathbb{C}^2$ with the standard inner product. Let A be a matrix in $M_2(\mathbb{C})$. As a unitary operator U in $M_2(\mathbb{C})$ acts isometrically and transitively on the unit ball of $\mathbb{C}^2$, and $\langle U^*AUx, x\rangle = \langle A(Ux), Ux\rangle$, we may conclude that $W(A) = W(U^*AU)$. By an appropriate choice of U, we may bring the matrix A to upper triangular form by a change of basis. So without loss of generality, we may assume that A is in upper-triangular form with the eigenvalues of A, λ_1, λ_2 on the diagonal and μ as the upper off-diagonal entry. For a unit vector $\widetilde{x} := [z_1 \;\; z_2]^t \in \mathbb{C}^2$ ($i.e. |z_1|^2 + |z_2|^2 = 1$), we see that $\langle A\widetilde{x}, \widetilde{x}\rangle = \lambda_1|z_1|^2 + \mu\overline{z_1}z_2 + \lambda_2|z_2|^2$. Thus the numerical range is an ellipse with foci at λ_1, λ_2 and minor axis of length $\sqrt{|\mu|}$. This proves that $W(A)$ is convex.

Let T be an operator in $\mathcal{B}(\mathscr{H})$. In the general case, for two unit vectors $\widetilde{x}, \widetilde{y}$ in $\mathscr{H}$, we would like to prove that any convex combination of $\langle T\widetilde{x}, \widetilde{x}\rangle$ and $\langle T\widetilde{y}, \widetilde{y}\rangle$ is in $W(T)$. If $\widetilde{x}, \widetilde{y}$ are scalar multiples of each other,$\langle T\widetilde{x}, \widetilde{x}\rangle = \langle T\widetilde{y}, \widetilde{y}\rangle$ and we are done. If they are linearly independent, we consider the two dimensional subspace V of $\mathscr{H}$ generated by $\widetilde{x}, \widetilde{y}$. If E denotes the orthogonal projection onto V, ETE can be represented as a 2×2 matrix in the basis $\{\widetilde{x}, \widetilde{y}\}$. Also $\langle T\widetilde{x}, \widetilde{x} = \langle ETE\widetilde{x}, \widetilde{x}\rangle$. But from what we have proved for matrices in $M_2(\mathbb{C})$), we get that $W(ETE)$ is convex. This finishes the proof.

$\square$

Remark 2.2.3. *Let $e_1, \cdots, e_n$ be the standard orthonormal basis vectors for $\mathbb{C}^n$ and A be a matrix in $M_n(\mathbb{C})$. Then the diagonal entries of A are $\langle Ae_1, e_1\rangle, \cdots, \langle Ae_n, e_n\rangle$ which lie in the numerical range of A.*

2.3 C^*-convexity

At this point, we direct the interested reader to [14] for an exposition on the basic notions and results in the theory of C^*-convexity, which may be though of as a noncommutative version of convexity. For the sake of completion, we include the necessary definitions below in order to be able to state Proposition 4.3.2.

Definition 2.3.1. *Let $\mathfrak{A}$ be a unital C^*-algebra with identity I. A set $\mathscr{S}$ in $\mathfrak{A}$ is said to be C^*-*convex *in $\mathfrak{A}$ if for operators $A_1, \cdots, A_n \in \mathscr{S}$ and operators $T_1, \cdots, T_n \in \mathfrak{A}$ satisfying $T_1^*T_1 + \cdots + T_n^*T_n = I$, the operator $T_1^*A_1T_1 + \cdots + T_n^*A_nT_n$, which is a C^*-convex combination of the A_i's, is also in $\mathscr{S}$.*

*We define the C^*-*segment *joining two operators $A_1, A_2 \in \mathfrak{A}$ to be the set $S(A_1, A_2) := \{T_1^*A_1T_1 + T_2^*A_2T_2 : T_1^*T_1 + T_2^*T_2 = I, T_1, T_2 \in \mathfrak{A}\}$. For operators $A_1, \cdots, A_n$ in $\mathfrak{A}$, the set $\{\sum_{i=1}^n T_i^*A_iT_i : \sum_{i=1}^n T_i^*T_i = I, T_1, \cdots, T_n \in \mathfrak{A}\}$ is called the C^*-polytope generated by $A_1, \cdots, A_n$.*

It is worthwhile to note that a C^*-segment (or C^*-polytope) need not even be convex, let alone C^*-convex. But in remark 6.2.6 for $M_n(\mathbb{C})$, we note that the set of spectral distributions of elements in a C^*-segment joining projections is a convex set.

2.4 The Schur-Horn theorem

Definition 2.4.1. *The permutation polytope generated by a vector $\tilde{x} = (x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^n$ denoted by $\mathcal{K}_{\tilde{x}}$ is defined as the convex hull of the set $\{(x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(n)}) \in \mathbb{R}^n : \pi \in S_n\}$. Here S_n denotes the symmetric group on $\{1, 2, \ldots, n\}$.*

The following lemma (Lemma 5 in [11]) characterizes the permutation polytope of a vector in $\mathbb{R}^n$.

Lemma 2.4.2. *If $x_1 \geq x_2 \geq \cdots \geq x_n, y_1 \geq y_2 \geq \cdots \geq y_n$, and $x_1 + x_2 + \cdots + x_n = y_1 + y_2 + \cdots + y_n$, then the following are equivalent :*

(i) $(y_1, y_2, \cdots, y_n)(= \tilde{y}) \in \mathcal{K}_{\tilde{x}}$.

(ii) $y_1 \leq x_1, y_1 + y_2 \leq x_1 + x_2, \ldots, y_1 + y_2 + \cdots + y_{n-1} \leq x_1 + x_2 + \cdots + x_n$

(iii) *There are points* $(x_1^{(1)}, x_2^{(1)}, \cdots, x_n^{(1)})(= \tilde{x}_1), \ldots, (x_1^{(n)}, x_2^{(n)}, \ldots, x_n^{(n)})(= \tilde{x}_n)$ *in* $\mathcal{K}_{\tilde{x}}$ *such that* $\tilde{x}_1 = \tilde{x}, \tilde{x}_n = \tilde{y}$, *and* $\tilde{x}_{k+1} = t\tilde{x}_k + (1-t)\tau(\tilde{x_k})$ *for each* k *in* $\{1, 2, \ldots, n-1\}$, *some transposition* τ *in* S_n, *and some* t *in* $[0, 1]$, *depending on* k.

The combined work of Schur in [16], and Horn in [7], completely characterizes the possible diagonals of a Hermitian matrix whose eigenvalues (with multiplicity) are known. Below we state and give a proof of the Schur-Horn theorem using Lemma 2.4.2.

Theorem 2.4.3. *(Schur-Horn) Let* $\mathbf{d} = \{d_i\}_{i=1}^N$ *and* $\lambda = \{\lambda_i\}_{i=1}^N$ *be real vectors. There is a Hermitian matrix with diagonal entries* $\{d_i\}_{i=1}^N$ *and eigenvalues* $\{\lambda_i\}_{i=1}^N$ *if and only if the vector* $(d_1, d_2, \ldots, d_n)$ *is in the permutation polytope generated by* $(\lambda_1, \lambda_2, \ldots, \lambda_n)$.

Proof. Let $A(= a_{jk})$ be a $n \times n$ Hermitian matrix with eigenvalues $\{\lambda_i\}_{i=1}^n$, counted with multiplicity. Denote the diagonal of A by $\tilde{a}$, thought of as a vector in $\mathbb{R}^n$, and the vector $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ by $\tilde{\lambda}$. Let Λ be the diagonal matrix having $\lambda_1, \lambda_2, \ldots, \lambda_n$ on its diagonal.

($\Rightarrow$) A may be written in the form $U\Lambda U^*$, where U is a unitary matrix. Then $a_{ii} = \sum_{j=1}^n \lambda_j |u_{ij}|^2,\ i = 1, 2, \ldots, n$. Let $S = (s_{ij})$ be the matrix defined by $s_{ij} = |u_{ij}|^2$.

Since U is a unitary matrix, S is a doubly stochastic matrix and we have $\tilde{a} = S\tilde{\lambda}$. By the Birkhoffvon Neumann theorem, S can be written as a convex combination of permutation matrices. Thus $\tilde{a}$ is in the permutation polytope generated by $\tilde{\lambda}$. This proves Schur's theorem.

($\Leftarrow$) If $\tilde{a}$ occurs as the diagonal of a Hermitian matrix with eigenvalues $\{\lambda_i\}_{i=1}^n$, then $t\tilde{a} + (1-t)\tau(\tilde{a})$ also occurs as the diagonal of some Hermitian matrix with the same set of eigenvalues, for any transposition τ in S_n. One may prove that in the following manner.Let ξ be a complex number of modulus 1 such that $\overline{\xi a_{jk}} = -\xi a_{jk}$ and U be a unitary matrix with $\xi\sqrt{t}, \sqrt{t}$ in the j,j and k,k entries, respectively, $-\sqrt{1-t^2}, \xi\sqrt{1-t^2}$ at the j,k and k,j entries, respectively, 1 at all diagonal entries other than j,j and k,k, and 0 at all other entries. Then UAU^* has $ta_{jj} + (1-t)a_{kk}$ at the j,j entry, $(1-t)a_{jj} + ta_{kk}$ at the k,k entry, and a_{ll} at the l,l entry where $l \neq j,k$. Let τ be the transposition of $\{1,2,\ldots,n\}$ that interchanges j and k. Then the diagonal of UAU^* is $t\tilde{a} + (1-t)\tau(\tilde{a})$. Λ is a Hermitian matrix with eigenvalues $\{\lambda_i\}_{i=1}^n$. Using the equivalence of (i) and (iii) in the lemma mentioned above, we see that any vector in the permutation polytope generated by $(\lambda_1, \lambda_2, \ldots, \lambda_n)$, occurs as the diagonal of a Hermitian matrix with the prescribed eigenvalues. This proves Horn's theorem.

$\square$

Remark 2.4.4. *An orthogonal projection is a Hermitian matrix with* $\widetilde{\lambda} := (1, \cdots, 1, 0, \cdots, 0)$ *as the vector of eigenvalues, where the number of* 1*'s is equal to the the rank,*

m, of the projection. The permutation polytope generated by λ is the set of all vectors $(d_1, \cdots, d_n)$ such that $0 \leq d_i \leq 1$ for $1 \leq i \leq n$, and $\sum_{i=1}^{n} d_i = m$, which, by the Schur-Horn theorem, give necessary and sufficient conditions for a vector in $\mathbb{R}^n$ to appear as the diagonal of a rank m projection.

Chapter 3

Some Technical Lemmas

In this chapter, we collect three technical lemmas and their proofs, that will be used in later chapters. The first section includes the proof of an inequality involving the entries of an orthogonal projection matrix in $M_n(\mathbb{C})$. This will be useful in Chapter 4 for error estimation in an algorithm to construct an orthogonal projection using a given vector in $\mathbb{R}^n$ prescribed as its diagonal entries. The second section includes a proof of the Douglas lemma for von Neumann algebras. We use it in Chapter 3 to prove some new results about the left (one-sided) ideals of von Neumann algebras. It is also used to prove an extension of a result on C^*-convexity, in [14]. In the third section, we prove a non-commutative version of the Gram-Schmidt process, for finite von Neumann algebras.

3.1 An Inequality

Lemma 3.1.1. *Let E be an orthogonal projection in $M_n(\mathbb{C})$ with (i,j)th entry denoted by a_{ij}. Then, for $i \neq j$, we have $|a_{ij}| \leq \min\{\sqrt{a_{ii}a_{jj}}, \sqrt{(1-a_{ii})(1-a_{jj})}\}$.*

Proof. Let $\{e_i\}_{i=1}^n$ denote the standard orthonormal basis for $\mathbb{C}^n$. We note that $a_{ij} = \langle Ee_i, Ee_j\rangle$. If $i \neq j$, as $\langle e_i, e_j\rangle = 0$, we also observe that $a_{ij} = \langle (I-E)e_i, (I-E)e_j\rangle$. By Cauchy-Schwartz inequality, $|a_{ij}| = |\langle Ee_i, Ee_j\rangle| \leq \|Ee_i\| \cdot \|Ee_j\| = \sqrt{a_{ii}} \cdot \sqrt{a_{jj}}$. Using the same argument for the projection $I - E$, we conclude that

$|a_{ij}| \leq \sqrt{(1-a_{ii})(1-a_{jj})}$. □

Remark 3.1.2. *In the above setting, if $a_{kk} = 1$ or 0 for some $k \in \{1, 2, \cdots, n\}$, then $a_{kj} = a_{jk} = 0$ for $1 \leq j \leq n, j \neq k$. In other words, if an orthogonal projection has 1 or 0 in the $(k,k)^{th}$ entry then the k^{th} column and k^{th} row must have all entries zero except possibly the $(k,k)^{th}$ entry.*

3.2 The Douglas Lemma

In [1], R. G. Douglas notes that the notions of majorization, factorization, and range inclusion, for operators on a Hilbert space are intimately connected. The main result of [1] (Theorem 1 in [1]) is referred to as "the Douglas lemma" or "the Douglas factorization theorem" in the literature. It naturally appears in many contexts. As Douglas observed, "fragments of these results are to be found scattered throughout the literature (usually buried in proofs) . . . ." In Proposition 3.3 of [15], G. K. Pedersen proves the following :

Proposition. If A is a 4-SAW^*-algebra, there is for each pair x, y in A such that $x^*x \leq y^*y$ an element w in A, with $\|w\| \leq 1$, such that $x = wy$.

This may be viewed as a generalization of the Douglas lemma to 4-SAW^*-algebras, relating majorization, and factorization, of operators in the 4-SAW^*-algebra A. Keeping in mind that every von Neumann algebra is a 4-SAW^*-algebra (in fact, a k-SAW^*-algebra for k in $\mathbb{N}$), as a corollary, one may note that the Douglas lemma holds true for von Neumann algebras.

As the Douglas lemma for von Neumann algebras will play a key role in our later applications, we formulate and give a proof of it below that does not depend on the proposition of Pedersen mentioned above.

3.2.1 A Proof of the Douglas lemma for von Neumann algebras

Theorem 3.2.1. *(Douglas factorization lemma) Let $\mathscr{R}$ be a von Neumann algebra acting on the Hilbert space $\mathscr{H}$. For A, B in $\mathscr{R}$ the following are equivalent :*

(i) $A^*A \leq \lambda^2 B^*B$ *for some* $\lambda \geq 0$*;*

(ii) $A = CB$ *for some operator* C *in* $\mathscr{R}$.

In addition, if $A^*A = B^*B$, then C can be chosen to be a partial isometry with initial projection the range projection of B, and final projection as the range projection of A.

Proof. ((i) $\Rightarrow$ (ii))

For any vector f in the Hilbert space $\mathscr{H}$, we have that $\|Af\|^2 = \langle A^*Af, f\rangle \leq \lambda^2\langle B^*Bf, f\rangle = \lambda^2\|Bf\|^2$ which implies $\|Af\| \leq \lambda\|Bf\|$. Thus if $Bf = 0$, it follows that $Af = 0$ and the linear map C defined on the range of B by $C(Bf) = Af$ is well-defined and also bounded (with norm less than λ). Thus we may extend the domain of definition of C to $\mathrm{ran}(B)^-$ the closure of the range of B. If h is a vector in $\mathrm{ran}(B)^\perp$, we define $Ch = 0$. Thus C is a bounded operator on $\mathscr{H}$ such that $A = CB$.

Let R be a self-adjoint operator in the commutant $\mathscr{R}'$ of $\mathscr{R}$. Then $RA = AR, RB = BR$ and the linear subspace $\mathrm{ran}(B)$ is invariant under R and so is the closed subspace $\mathrm{ran}(B)^\perp$ (as R is self-adjoint). For vectors f_1 in $\mathscr{H}$ and f_2 in $\mathrm{ran}(B)^\perp$, we have that $CR(Bf_1 + f_2) = CRBf_1 + C(Rf_2) = CB(Rf_1) + 0 = A(Rf_1) = R(Af_1) = RCBf_1 = RC(Bf_1 + f_2)$. Thus RC and CR coincide on the dense subspace of $\mathscr{H}$ given by $\mathrm{ran}(B) \oplus \mathrm{ran}(B)^\perp$. Being bounded operators, we note that $RC = CR$ for any self-adjoint operator R in $\mathscr{R}'$. As every element in a von Neumann algebra can be written as a finite linear combination of self-adjoint elements, we conclude that C commutes with every element in $\mathscr{R}'$. By the double commutant theorem, C is in $(\mathscr{R}')' = \mathscr{R}$.

((ii) $\Rightarrow$ (i))

If $A = CB$ for some operator $C \in \mathscr{R}$, then $A^*A = B^*C^*CB \leq \|C\|^2 B^*B$. Thus , we may pick $\lambda = \|C\|$.

If $A^*A = B^*B$, then $\|Af\| = \|Bf\|$ for any vector f in $\mathscr{H}$. Thus, the second part follows from the explicit definition of the operator C earlier in the proof. □

The polar decomposition theorem for von Neumann algebras is a direct consequence of the Douglas lemma.

Corollary 3.2.2. *(Polar decomposition theorem) Let $\mathscr{R}$ be a von Neumann algebra acting on the Hilbert space $\mathscr{H}$. For an operator A in $\mathscr{R}$, there is a partial isometry V with initial projection the range projection of $(A^*A)^{\frac{1}{2}}$, and final projection as the range projection of A such that $A = V(A^*A)^{\frac{1}{2}}$.*

Proof. Let B denote the operator $(A^*A)^{\frac{1}{2}}$. Clearly $A^*A = B^*B$ and thus from the second part of the Douglas lemma, the corollary follows. □

3.3 A Non-commutative Gram-Schmidt Process

The Gram-Schmidt process is used to obtain an orthonormal basis for a finite-dimensional Hilbert space (say $\mathbb{C}^n, n \in \mathbb{N}$ with the standard inner product) from a given basis of the Hilbert space. One may use it to construct an orthonormal basis for the Hilbert space starting with a prescribed unit vector. Sometimes we are more interested in extracting the change of basis matrix from the standard orthonormal basis in $\mathbb{C}^n$ to the orthonormal basis constructed containing the initial unit vector.

With this viewpoint, we may consider it as a reconstruction algorithm to generate a unitary matrix in $M_n(\mathbb{C})$ where the first row is known (which is necessarily a unit vector in $(\mathbb{C}^n)^*$.)

In this section, we formulate and prove a non-commutative version of the Gram-Schmidt process. Along with Kadison's diagonalization theorem ([9]), this will be used as a tool to prove results about C^*-convexity. Let $\mathscr{R}$ be a von Neumann algebra and $T_1, T_2, \cdots, T_n$ be elements of $\mathscr{R}$ such that $\sum_{i=1}^n T_i T_i^* = 1_{\mathscr{R}}$. The following question naturally arises in this context : Is there always a unitary operator in $M_n(\mathscr{R})$ with $[T_1 \quad T_2 \quad \cdots \quad T_n]$ as its first row ? As we will see below, the answer is in the affirmative if $\mathscr{R}$ is finite but not otherwise.

For $T \in M_n(\mathscr{R})$, we denote its (i,j) entry by T_{ij}. Although Proposition 2.3.1 and Lemma 2.3.2 below are well-known, we include proofs for the sake of completion.

Proposition 3.3.1. *Let $\mathscr{R}$ be a finite von Neumann algebra. Then for $n \in \mathbb{N}, M_n(\mathscr{R})$ is also a finite von Neumann algebra.*

Proof. There is a natural diagonal embedding ι of $\mathscr{R}$ in $M_n(\mathscr{R})$ which sends $A \in \mathscr{R}$ to $diag(A, \cdots, A) \in M_n(\mathscr{R})$. The embedding ι sends elements in the center $\mathscr{C}$ of $\mathscr{R}$ to elements in the center of $M_n(\mathscr{R})$. In fact, the center of $M_n(\mathscr{R})$ is $\iota(\mathscr{C})$.

Let τ be the unique center-valued trace on $\mathscr{R}$. Then for $A \in M_n(\mathscr{R})$ define a center-valued linear map Tr in the following way,

$$Tr(A) := \frac{1}{n}\sum_{i=1}^{n} \iota(\tau(A_{ii})) = \frac{1}{n}\iota(\tau(\sum_{i=1}^{n} A_{ii}))$$

where A_{ij} denotes the (i,j)th entry of A..

Let $A, B \in M_n(\mathscr{R})$. We will prove that Tr is a center-valued trace for $M_n(\mathscr{R})$.

(i)

$$\begin{aligned} Tr(AB) &= \frac{1}{n}\iota(\tau(\sum_{i=1}^{n}(AB)_{ii})) = \frac{1}{n}\iota(\tau(\sum_{i=1}^{n}\sum_{j=1}^{n}A_{ij}B_{ji})) \\ &= \iota(\tau(\sum_{j=1}^{n}\sum_{i=1}^{n}B_{ji}A_{ij})) = \frac{1}{n}\iota(\tau(\sum_{i=1}^{n}(BA)_{ii})) \\ &= Tr(BA) \end{aligned}$$

(ii) Let C be an element in the center of $M_n(\mathscr{R})$. Then $C = \iota(z)$ for some $z \in \mathscr{C}$ and $\tau(z) = z$. We have that $Tr(C) = \frac{1}{n}\iota(\tau(\sum_{i=1}^{n} C_{ii})) = \frac{1}{n}\iota(\tau(nz)) = \iota(\tau(z)) = \iota(z) = C$

(iii) Let $A > 0$ in $M_n(\mathscr{R})$. Then $A = BB^*$ for some $B \neq 0, \in M_n(\mathscr{R})$. Thus for some index (i,j) we have that $\tau(B_{ij}B_{ij}^*) > 0$. Thus $Tr(A) = Tr(BB^*) = \frac{1}{n}\iota(\tau(\sum_{i=1}^{n}\sum_{j=1}^{n} B_{ij}B_{ij}^*)) > 0$

Thus $M_n(\mathscr{R})$ is a finite von Neumann algebra as it has a center-valued trace, namely Tr. □

Lemma 3.3.2. *Let E and F be two (Murray-von Neumann) equivalent projections in a finite von Neumann algebra. Then the projections $I - E$ and $I - F$ are also equivalent.*

Proof. Let's assume, on the contrary, that $I - E$ and $I - F$ are not equivalent. Then there exists a central projection P such that either $P(I-E) \prec P(I-F)$ or $P(I-F) \prec P(I-E)$. Without loss of generality, we may assume that the former holds.

Let Q be a projection such that $P(I-E) \sim Q < P(I-F)$. As $E \sim F$, we have that $PE \sim PF$ as P is a central projection. Thus, $P = P(I-E) + PE \sim Q + PF \prec P(I-F) + PF = P$. This is a contradiction as P is a projection in a finite von Neumann algebra. This proves that $I - E$ and $I - F$ are equivalent. $\square$

Theorem 3.3.3. *If $T_i, i \in \{1, 2, \cdots, n\}$ are n elements in a finite von Neumann algebra $\mathscr{R}$ and $\sum_{i=1}^{n} T_i T_i^* = 1_{\mathscr{R}}$, then there is a unitary operator U in $M_n(\mathscr{R})$ whose first row is $[T_1 \ \ T_2 \ \ \cdots \ \ T_n]$.*

Proof. Let V (in $M_n(\mathscr{R})$) be given by

$$\begin{bmatrix} T_1 & T_2 & T_3 & \dots & T_n \\ 0 & 0 & 0 & \dots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We have that

$$VV^* = \begin{bmatrix} 1_{\mathscr{R}} & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Let E, F be defined as VV^*, V^*V respectively. Note that E is a projection. Thus, V is a partial isometry with initial projection F and final projection E. As $M_n(\mathscr{R})$ is finite, by Lemma 2, there is a partial isometry W with initial projection $I - F$ and final projection $I - E$, where I denotes the identity element in $M_n(\mathscr{R})$.

As the final projection of W^* (i.e. $I - F$) is orthogonal to the initial projection of V (i.e. F), we observe that $VW^* = 0$ and $(V + W)(V + W)^* = I$. With $M_n(\mathscr{R})$ being a finite von Neumann algebra, this implies that $V + W$ is a unitary operator.

As $WW^* = I - E$, the $(1, 1)$ entry of WW^* is 0. Thus $\sum_{i=1}^{n} W_{1i}W_{1i}^* = 0$ which implies that $W_{1i} = 0$ for $i \in \{1, 2, \cdots, n\}$. This means that the first row of W is $[0\ 0\ \cdots\ 0]$. We conclude that $U := V + W$ is a unitary operator with first row $[T_1\ \ T_2\ \ \cdots\ \ T_n]$.

$\square$

The assumption in the above theorem that the von Neumann algebra $\mathscr{R}$ be finite cannot be dispensed with, as is clear from the remark below.

Remark 3.3.4. *Let $\mathscr{R}$ be a properly infinite von Neumann algebra. Then for $i \in \{1, 2, \cdots, n\}$ there exist partial isometries V_i with initial projection $1_{\mathscr{R}}$ and final projection P_i such that $\sum_{i=1}^{n} P_i = 1_{\mathscr{R}}$ and $P_i \sim P_j, 1 \le i, j \le n$. In other words,*

$\sum_{i=1}^n V_iV_i^* = 1_{\mathscr{R}}, V_i$'s are isometries and for $i \neq j, V_j^*V_i = 0$. If W_i in $\mathscr{R}$ for $1 \leq i \leq n$ are such that $\sum_{i=1}^n W_iW_i^* = 1_{\mathscr{R}}$, then not all of the W_i's can be 0. As $V_j^*(\sum_{i=1}^n V_iW_i^*) = W_j^*$, which is non-zero for some $j \in \{1, 2, \cdots, n\}$, we have that $\sum_{i=1}^n V_iW_i^* \neq 0$. Thus although $\sum_{i=1}^n V_iV_i^* = 1_{\mathscr{R}}$, there is no unitary operator in $M_n(\mathscr{R})$ whose first row is $[V_1 \;\; V_2 \;\; \cdots \;\; V_n]$.

Chapter 4

Some Applications of the Douglas Lemma

The fact that $\mathcal{B}(\mathscr{H})$ has a norm and an involutive, norm-preserving adjoint operation inherited from its (continuous) action on $\mathscr{H}$, patently, has serious consequences for its metric and geometric structures, but these consequences extend, as well, to its basic algebraic structure. In particular, the ideal structure of a C*-algebra is affected by this addition of structures on $\mathcal{B}(\mathscr{H})$. Of course, the (generally) infinite (linear-) dimensionality of $\mathcal{B}(\mathscr{H})$ shifts the study to infinite-dimensional algebras — a not very congenial topic in algebra (largely stemming from the fact that there are self-adjoint operators in $\mathcal{B}(\mathscr{H})$ with only 0 in their null spaces and ranges that are dense in but not all of $\mathscr{H}$) Still, each proper (left, right, and two-sided) ideal $\mathscr{I}$ in a C*-subalgebra $\mathfrak{A}$ of $\mathcal{B}(\mathscr{H})$ has a norm-closure in $\mathfrak{A}$ that is, again, a proper (norm-closed) ideal in$\mathfrak{A}$.

Hence, using Zorn's Lemma, $\mathscr{I}$ is contained in a proper, maximal (left, right, or two-sided) ideal in $\mathfrak{A}$ that is proper and norm closed.

Drawing from [G-N], Segal constructed a representation of a C*-algebra $\mathfrak{A}$ associated with a special class of (complex-)linear functionals on $\mathfrak{A}$ known as *states* of $\mathfrak{A}$. A state ρ of $\mathfrak{A}$ is defined as a functional on $\mathfrak{A}$ satisfying $\rho(I) = 1$ and $\rho(A^*A) \geq 0$ for each A in $\mathfrak{A}$. Roughly speaking, $\langle A, B\rangle_\rho =: \rho(B^*A)$ defines a positive, semi-definite inner product on $\mathfrak{A}$. When we "divide out" by the set $\mathcal{N}_\rho$ of "null vectors" in $\mathfrak{A}$ (those A such that $\rho(A^*A) = 0$), that is, consider $\mathfrak{A}$ modulo $\mathcal{N}_\rho$, the resulting linear space, $\mathfrak{A}/\mathcal{N}_\rho$, inherits a positive definite inner product from the ρ-inner product on $\mathfrak{A}$.) The completion, $\mathscr{H}_\rho$ of $\mathfrak{A}/\mathcal{N}_\rho$ relative to this positive definite inner product is the Hilbert space on which the GNS representation, π_ρ of $\mathfrak{A}$ (associated with ρ) takes place. The analytic details of this process involves creative use of ρ in the setting of full application of the Cauchy-Schwarz inequality. One consequence of these Cauchy-Schwarz calculations is that $\mathcal{N}_\rho$ is a left ideal in $\mathfrak{A}$. Hence, the "left action," by $\mathfrak{A}$ on the quotient vector space $\mathfrak{A}/\mathcal{N}_\rho$, given by $\pi_\rho(A)(B + \mathcal{N}_\rho) = AB + \mathcal{N}_\rho$, is well-defined (as $\pi_\rho(A)$ maps $\mathcal{N}_\rho$ into $\mathcal{N}_\rho$). Segal [17] shows that this GNS representation , π_ρ, is irreducible (topologically, that is, there are no closed subspaces of $\mathscr{H}_\rho$ stable under all $\pi_\rho(A)$ other than (0) and $\mathscr{H}_\rho$ if and only if ρ is a *pure* state of $\mathfrak{A}$ (an extreme point of the convex set of states of $\mathfrak{A}$). It is shown in [8] that an irreducible representation, π, of a C*-algebra, $\mathfrak{A}$ on a Hilbert space, $\mathscr{H}$, is "transitive." (Given linearly independent vectors, $x_1, \ldots, x_n$, and any n other vectors $y_1, \ldots, y_n$, there is an A in $\mathfrak{A}$ such that $\pi(A)x_1 = y_1, \ldots, \pi(A)x_n = y_n$. From this, π is algebraically irreducible (no proper subspaces, closed or not, stable under all $\pi(A)$.) It follows that the ideal $\mathcal{N}_\rho$ (the "left kernel" of ρ) is maximal (necessarily, proper and closed) if and only ρ is pure. The adjoint $\mathcal{N}_\rho^*$ of $\mathcal{N}_\rho$ is a maximal right ideal in $\mathfrak{A}$, and $\mathcal{N}_\rho + \mathcal{N}_\rho^*$ $(= \{A + B : \, A \in \mathcal{N}_\rho, \, B \in \mathcal{N}_\rho^*\})$ is the null space of ρ in this case, as shown in [8].

As we see, the structures of the left ideals and right ideals in a C*-algebra, $\mathfrak{A}$, are very closely tied to the representation theory of $\mathfrak{A}$. In particular, the representation theory of $\mathcal{B}(\mathscr{H})$ is very much a part of this. Coupled with Glimm's work in [2], this approach, applied to one of the Glimm algebras, the CAR algebra, these considerations complete the theoretical foundations of he study of representations of the Canonical Anti-commutation Relations. These are of great interest in quantum statistical mechanics which provide us with added motivation for gathering such information as we can about ideal structure in von Neumann and C*-algebras.

4.1 The Douglas property and polar decomposition

Definition 4.1.1. *Let $\mathfrak{A}$ be a C^*-algebra. We say that $\mathfrak{A}$ satisfies the Douglas property (DP) if for any two elements of A, B of $\mathfrak{A}$, the following are equivalent :*

(i) $A^*A \leq B^*B$;

(ii) $A = CB$ *for some operator* C *in* $\mathfrak{A}$.

Note that here we normalize the positive constant λ in the Douglas lemma (by absorbing it in B).

On a similar note, we say that $\mathfrak{A}$ satisfies the weak polar decomposition property *(WPDP) if for any element A in $\mathfrak{A}$, there is an operator V in $\mathfrak{A}$ such that $A = V(A^*A)^{\frac{1}{2}}$. We caution the reader to note that we do not require that V be a partial isometry.*

Example 4.1.2. *Let $X := \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\} \subset \mathbb{R}$. Note that X is a compact Hausdorff space in the topology inherited from $\mathbb{R}$. Let f, g be two functions in the C^*-algebra $C(X;\mathbb{C})$ defined as follows :*

$$f(x) = \begin{cases} x & \text{if } x = \frac{1}{2n}\ (n \in \mathbb{N}) \\ 0 & \text{otherwise} \end{cases}$$

and $g(x) = x$ for all x in X.

Clearly, $\overline{f}f = f^2 \leq g^2 = \overline{g}g$. If h is a complex-valued function such that $f = hg$, for $n \in \mathbb{N}$ we must have $h(x_n) = 1$ where $x_n = \frac{1}{2n}$, and $h(y_n) = 0$ where $y_n = \frac{1}{2n-1}$. But as,

$$\lim_{n\to\infty} x_n = 0, \lim_{n\to\infty} h(x_n) = 1 \text{ and } \lim_{n\to\infty} y_n = 0, \lim_{n\to\infty} h(y_n) = 0,$$

h cannot be continuous. Thus $C(X;\mathbb{C})$ does not satisfy the Douglas property.

Theorem 4.1.3. *A commutative C^*-algebra $\mathfrak{A}$ satisfies the Douglas property if and only if it satisfies the weak polar decomposition property.*

Proof. Let X denote the Gelfand space of $\mathfrak{A}$ i.e. the space of characters on $\mathfrak{A}$ with the weak-$*$ topology. From the Gelfand representation, we know that $\mathfrak{A}$ is $*$-isomorphic to $C_0(X;\mathbb{C})$, the set of complex-valued continuous functions on X vanishing at infinity (with complex conjugation as the involution). In the proof, we use the function-representation for the operators.

((DP) $\Rightarrow$ (WPDP))

Let $f \in C_0(X;\mathbb{C})$. Thus $g := |f|$ is also in $C_0(X;\mathbb{C})$. As $|f|^2 = |g|^2$ (and hence, $\overline{f}f \le \overline{g}g = |f|^2$), by the Douglas property we must have a function h in $C_0(X;\mathbb{C})$ such that $f = hg = h|f|$. Thus $\mathfrak{A}$ satisfies the weak polar decomposition property.

((WPDP) $\Rightarrow$ (DP))

Let $f, g \in C_0(X;\mathbb{C})$ such that $|f|^2 = \overline{f}f \le \overline{g}g = |g|^2$ (i.e. $|g|^2 - |f|^2 \ge 0$). Consider the complex-valued continuous function defined by $s := |f| + \imath\sqrt{|g|^2 - |f|^2}$. Clearly s is in $C_0(X;\mathbb{C})$ as $|f|, |g|$ are in $C_0(X;\mathbb{C})$. As $\mathfrak{A}$ satisfies WPDP, we have a continuous function v such that $s = v|s|$. Let $v = v_1 + \imath v_2$ where v_1, v_2 are real-valued continuous functions in $C_0(X;\mathbb{C})$. Note that $s = v_1|s| + \imath v_2|s|$ and $|s| = \sqrt{|f|^2 + (|g|^2 - |f|^2)} = \sqrt{|g|^2} = |g|$. Comparing the real parts we observe that $\mathrm{Re}(s) = |f| = v_1|g|$. Again using polar decomposition for f, g we have functions v_f, v_g in $C_0(X;\mathbb{C})$ such that $f = v_f|f|, g = v_g|g|$. Let $\mathfrak{Z}(g)(:= \{x \in X : g(x) = 0\})$ denote the zero-set of g. On the co-zero set of g (i.e. $\mathfrak{Z}(g)^c$), v_g takes values in the unit circle in $\mathbb{C}$. As the zero-sets of g and $|g|$ coincide i.e. $\mathfrak{Z}(g) = \mathfrak{Z}(|g|)$, we conclude that $|g| = \overline{v_g}g$ and clearly $\overline{v_g}$ is in $C_0(X;\mathbb{C})$. Thus from the equation $|f| = v_1|g|$, we have that $f = (v_f v_1 \overline{v_g})g = hg$

for some continuous function $h = v_f v_1 \overline{v_g}$ in $C_0(X;\mathbb{C})$. Thus $\mathfrak{A}$ satisfies the Douglas property. $\square$

Remark 4.1.4. *Let Y be a closed subset of a locally compact Hausdorff space X. We denote the one-point compactification of X by $X^* := X \cup \{\infty\}$ where ∞ is the point at infinity (distinct from the points of X). We note that $Y \cup \{\infty\}$ is closed in X^*, as for compact Y, the closure in X^* is Y itself and for non-compact Y, the closure is $Y \cup \{\infty\}$.*

Let us endow Y with the subspace topology inherited from X. We may extend a function f in $C_0(Y;\mathbb{C})$ to a continuous function on the compact set $Y \cup \{\infty\}$ in X^*, by defining $f(\infty) = 0$. By the Tietze extension theorem, there is a continuous extension $\widetilde{f}$ of f to the whole of X^*. As $\widetilde{f}(\infty) = 0$, we note that $\widetilde{f}|_X$ is in $C_0(X;\mathbb{C})$ and restricts to f on Y. Thus, every function f in $C_0(Y;\mathbb{C})$ has an extension to a function in $C_0(X;\mathbb{C})$.

Remark 4.1.5. *Let Y be a closed subset of a locally compact Hausdorff space X which satisfies the Douglas property. Then $C_0(Y;\mathbb{C})$ also satisfies the Douglas property. We elaborate on this below.*

Let f be a function in $C_0(Y;\mathbb{C})$. By remark 4.1.4, we may extend f to a function $\widetilde{f}$ in $C_0(X;\mathbb{C})$. From theorem 4.1.3, we observe that there is a function $\widetilde{v}$ in $C_0(X;\mathbb{C})$ such that $\widetilde{f} = \widetilde{v}|\widetilde{f}|$. As $v(\infty) = 0$ in the extension to X^*, clearly v, the restriction of $\widetilde{v}$ to Y is in $C_0(Y;\mathbb{C})$ and $f = v|f|$. Thus $C_0(Y;\mathbb{C})$ satisfies the Douglas property.

Theorem 4.1.6. *Let X be a locally compact Hausdorff space. The C^*-algebra*

$C_0(X;\mathbb{C})$ satisfies the Douglas property if and only if X is sub-Stonean.

Proof. Let $C_0(X;\mathbb{C})$ satisfy the Douglas property. Let U, V be disjoint σ-compact open sets in X. Let f be a real-valued function in $C_0(X;\mathbb{C})$ such that $U = \{x \in X : f(x) > 0\}, V = \{x \in X : f(x) < 0\}$. As $C_0(X;\mathbb{C})$ satisfies the Douglas property, there is a function v_f in $C_0(X;\mathbb{C})$ such that $f = v_f|f|$. Note that $v_f \equiv 1$ on U (and thus, $\overline{U}$) and $v_f \equiv -1$ on V (and thus, $\overline{V}$). Hence $\overline{U} \cap \overline{V} = \varnothing$. Also the sets $\overline{U}, \overline{V}$ are compact as v_f vanishes at infinity. Thus X is sub-Stonean.

Next we prove the converse. Let X be sub-Stonean. Given a function f in $C_0(X;\mathbb{C})$, the set $U := \{x \in X : f(x) \neq 0\}$ i.e. the co-zero set of f is a σ-compact set, with compact closure. Let $v_f : U \to S^1$ be the continuous function defined by $v_f(x) = \frac{f(x)}{|f(x)|}$ for x in U. As S^1 is compact, by Corollary 1.11 in [4], v_f may be extended to a function from $\overline{U}$ to S^1. Then by remark 4.1.4, one may extend v_f to a function in $C_0(X;\mathbb{C})$ and we have $f = v_f|f|$. Thus $C_0(X;\mathbb{C})$ satisfies the Douglas property.

□

4.2 Left ideals of von Neumann algebras

Lemma 4.2.1. *Let A, B be operators in a von Neumann algebra $\mathscr{R}$. Then the left ideal $\mathscr{R}A$ is contained in the left ideal $\mathscr{R}B$ if and only if $A^*A \leq \lambda^2 B^*B$ for some $\lambda \geq 0$. As a consequence, for any A in $\mathscr{R}$, we have that $\mathscr{R}A = \mathscr{R}\sqrt{A^*A}$.*

Proof. Its straightforward to see that A is in $\mathscr{R}B$ if and only if $\mathscr{R}A \subseteq \mathscr{R}B$. And from Theorem 3.2.1, we have that A is in $\mathscr{R}B$ if and only if $A^*A \leq \lambda^2 B^*B$ for some $\lambda \geq 0$.

Further, $\mathscr{R}A = \mathscr{R}B$ if and only if $B^*B \leq \lambda^2 A^*A$ and $A^*A \leq \mu^2 B^*B$ for some $\lambda, \mu \geq 0$. In particular, if $A^*A = B^*B$, then $\mathscr{R}A = \mathscr{R}B$. Noting that $A^*A = \sqrt{A^*A}\sqrt{A^*A}$, we conclude that $\mathscr{R}A = \mathscr{R}\sqrt{A^*A}$.

□

Lemma 4.2.2. *Let A be an operator in a von Neumann algebra $\mathscr{R}$. Then the left ideal $\mathscr{R}A$ is weak-operator closed if and only if 0 is an isolated point in the spectrum of A^*A (and hence, $\sqrt{A^*A}$).*

Proof. If $A = 0$, the conclusion is straightforward. So we may assume that $A \neq 0$ and thus $\{0\}$ is a proper subset of the spectrum of the self-adjoint operator A^*A.

If $\mathscr{R}A$ is weak-operator closed, there is a unique projection E in $\mathscr{R}$ such that $\mathscr{R}A = \mathscr{R}E$. From Lemma 4.2.1, there are $\mu, \lambda > 0$ such that $\mu^2 E \leq A^*A \leq \lambda^2 E$. This tells us that the spectrum of A^*A is contained in $\{0\} \cup [\mu, \lambda]$ which implies that 0 is an isolated point in the spectrum of A^*A.

For the converse, let 0 be an isolated point in the spectrum of A^*A. By the spectral mapping theorem, 0 is also an isolated point in the spectrum of $\sqrt{A^*A}$. Let the distance of 0 from $\mathrm{sp}(\sqrt{A^*A}) - \{0\}$ (which is compact as 0 is isolated) be $\mu > 0$ and $\lambda = \|A\|$. Let F be the largest projection in $\mathscr{R}$ such that $\sqrt{A^*A}F = 0$. Then we have that $\mu^2(I - F) \leq A^*A \leq \lambda^2(I - F)$. Thus $\mathscr{R}A = \mathscr{R}(I - F)$ which is weak-operator closed. □

Proposition 4.2.3. *Let A be an operator in a von Neumann algebra $\mathscr{R}$ acting on the Hilbert space $\mathscr{H}$. Then the left ideal $\mathscr{R}A$ is norm-closed if and only if $\mathscr{R}A$ is weak-operator closed.*

Proof. Let $\mathscr{R}A$ be norm-closed. Without loss of generality, we may assume that A is positive (as $\mathscr{R}A = \mathscr{R}\sqrt{A^*A}$). By the Stone-Weierstrass theorem, we have that for a continuous function f on the spectrum of A vanishing at 0, $f(A)$ is in $\mathscr{R}A$. In particular, $\sqrt{A}$ is in $\mathscr{R}A$. Thus there is a $\lambda > 0$ such that $(\sqrt{A})^2 = A \leq \lambda^2 A^2$. The operator $\lambda^2 A^2 - A$ is positive and by the spectral mapping theorem has spectrum $\{\lambda^2\mu^2 - \mu : \mu \in \mathrm{sp}(A)\}$. For a non-zero element μ in the spectrum of A, $\lambda^2\mu^2 - \mu \geq 0 \Rightarrow \mu \geq \frac{1}{\lambda^2}$. This tells us that 0 is an isolated point in the spectrum of A and hence $\mathscr{R}A$ is weak-operator closed.

The converse is straightforward as the weak-operator topology on $\mathscr{R}$ is coarser than the norm topology. □

Let $\mathscr{R}$ be a von Neumann algebra. We use the notation $\langle V \rangle$, to denote the linear span of a subset V of $\mathscr{R}$.

Definition 4.2.4. *Let $\mathcal{S}$ be a family of operators in the von Neumann algebra $\mathscr{R}$. The smallest left ideal of $\mathscr{R}$ containing $\mathcal{S}$ is denoted by $\langle \mathscr{R}\mathcal{S} \rangle$ and said to be generated by $\mathcal{S}$. A left ideal $\mathscr{I}$ is said to be finitely generated (countably generated) if $\mathscr{I} = \langle \mathscr{R}\mathcal{S} \rangle$ for a finite (countable) subset $\mathcal{S}$ of $\mathscr{R}$. Here we take a moment to stress that the set of generators is considered in a purely algebraic sense.*

Proposition 4.2.5. *Let* A_1, A_2 *be operators in a von Neumann algebra* $\mathscr{R}$*. Then* $\mathscr{R}A_1 + \mathscr{R}A_2 = \mathscr{R}\sqrt{A_1^*A_1 + A_2^*A_2}$*. Thus, every finitely generated left ideal of* $\mathscr{R}$ *is a principal ideal.*

Proof. Consider the operators $A, \widetilde{A}$ in $M_2(\mathscr{R})$ represented by,

$$A = \begin{bmatrix} A_1 & 0 \\ A_2 & 0 \end{bmatrix}, \widetilde{A} = \begin{bmatrix} \sqrt{A_1^*A_1 + A_2^*A_2} & 0 \\ 0 & 0 \end{bmatrix}$$

It is easy to see that $A^*A = \widetilde{A}^*\widetilde{A}$. By Lemma 4.2.1, $M_2(\mathscr{R})A = M_2(\mathscr{R})\widetilde{A}$ and comparing the $(1,1)$ entry on both sides, our result follows.

Inductively, we see that for operators $A_1, \ldots, A_n$ in $\mathscr{R}$, $\mathscr{R}A_1 + \cdots + \mathscr{R}A_n = \mathscr{R}\sqrt{A_1^*A_1 + \cdots + A_n^*A_n}$. In conclusion, every finitely generated left ideal of $\mathscr{R}$ is singly generated. □

Corollary 4.2.6. *If* $\mathscr{I}$ *is a norm-closed left ideal of* $\mathscr{R}$ *which is finitely generated, then* $\mathscr{I}$ *is weak-operator closed.*

Proof. A straightforward consequence from Proposition 4.2.3, 4.2.5. □

Theorem 4.2.7. *If* $\mathscr{I}$ *is a norm-closed left ideal of* $\mathscr{R}$ *which is countably generated, then* $\mathscr{I}$ *is weak-operator closed (and thus, a principal ideal).*

Proof. Let $\mathscr{I}$ be a countably generated norm-closed left ideal of $\mathscr{R}$ with generating set $\mathcal{S} := \{A_i : i \in \mathbb{N}\}$. We prove that it must be weak-operator closed. Noting that $\mathscr{R}A_i = \mathscr{R}\sqrt{A_i^*A_i}$ and after appropriate scaling, we may assume that the A_i's are positive contractions (i.e. A_i's are positive and $\|A_i\| \le 1$). For $n \in \mathbb{N}$, define $B_n := \sqrt{\sum_{i=1}^{n} \frac{A_i^2}{2^n}}$. Thus the sequence $\{B_i^2\}_{i=1}^{\infty}$ is an increasing Cauchy sequence of positive operators in $\mathscr{I}$ and $\lim_{n\to\infty} B_n^2$ is a positive operator B^2 in $\mathscr{I}$. As $\mathscr{I}$ is norm-closed, B, the positive square-root of B^2 is in $\mathscr{I}$ and thus $\mathscr{R}B \subseteq \mathscr{I}$. Also for each $n \in \mathbb{N}$ as $A_n^2 \le 2^n B_n^2 \le 2^n B^2$, by Lemma 4.2.1, we have that $\mathscr{R}A_n \subseteq \mathscr{R}B$. Thus $\mathscr{I} \subseteq \mathscr{R}B$ and combined with the previous conclusion, $\mathscr{I} = \mathscr{R}B$. By Corollary 4.2.6, being norm-closed, $\mathscr{I} = \mathscr{R}B$ is also weak-operator closed.

□

Below we note a result about norm-closed left ideals of represented C*-algebras. In the results that follow after, we will see how a similar conclusion holds for left ideals in von Neumann algebras.

Proposition 4.2.8. *Let $\mathfrak{A}$ be a C*-algebra acting on the Hilbert space $\mathscr{H}$ and let $\mathscr{I}$ be a norm-closed left ideal of $\mathfrak{A}$. Then there is a norm-closed left ideal $\mathscr{J}$ of $\mathcal{B}(\mathscr{H})$ such that $\mathscr{I} = \mathscr{J} \cap \mathfrak{A}$.*

Proof. For a state ρ on a C^*-algebra, we denote its left kernel, as defined in [13] (Section 4.5.2), by $\mathcal{L}_\rho$. Let $\mathscr{P}^I$ denote the set of pure states on $\mathfrak{A}$ whose left kernels contain $\mathscr{I}$. Then from Theorem 3.2 in [8], we have that

$$\mathscr{I} = \bigcap_{\rho \in \mathscr{P}^I} \mathcal{L}_\rho$$

A pure state ρ on $\mathfrak{A}$ can be extended to a pure state $\overline{\rho}$ on $\mathcal{B}(\mathscr{H})$. We denote the set of all such extensions of the states in $\mathscr{P}^I$ by $\overline{\mathscr{P}}^I$. Being an intersection of norm-closed left ideals, the set

$$\mathscr{J} := \bigcap_{\overline{\rho} \in \overline{\mathscr{P}}^I} \mathcal{L}_{\overline{\rho}}$$

is also a norm-closed left ideal of $\mathcal{B}(\mathscr{H})$. Clearly if $\overline{\rho}$ in $\overline{\mathscr{P}}^I$ is an extension of a state ρ in $\mathscr{P}^I$, we have that $\mathcal{L}_{\overline{\rho}} \cap \mathfrak{A} = \mathcal{L}_\rho$. Thus we conclude that $\mathscr{I} = \mathscr{J} \cap \mathfrak{A}$. $\square$

Proposition 4.2.9. *Let $\mathscr{R}_1, \mathscr{R}_2$ be von Neumann algebras acting on the Hilbert space $\mathscr{H}$. Let A be an operator in $\mathscr{R}_1 \cap \mathscr{R}_2$. Then $\mathscr{R}_1 A \cap \mathscr{R}_2 = \mathscr{R}_1 A \cap \mathscr{R}_2 A = (\mathscr{R}_1 \cap \mathscr{R}_2)A$.*

Proof. Let B be an operator in $\mathscr{R}_1 A \cap \mathscr{R}_2$. As $B \in \mathscr{R}_1 A$, we have that $B^*B \leq \lambda^2 A^*A$ for some $\lambda \geq 0$. As B, A are both in $\mathscr{R}_2$, we conclude from the Douglas factorization lemma that B is also in $\mathscr{R}_2 A$. Thus $B \in \mathscr{R}_1 A \cap \mathscr{R}_2 A$. This proves that $\mathscr{R}_1 A \cap \mathscr{R}_2 \subseteq \mathscr{R}_1 A \cap \mathscr{R}_2 A$. The reverse inclusion is obvious. Thus $\mathscr{R}_1 A \cap \mathscr{R}_2 = \mathscr{R}_1 A \cap \mathscr{R}_2 A$.

By considering the von Neumann algebra $\mathscr{R}_1 \cap \mathscr{R}_2$ (in place of $\mathscr{R}_2$), we have from the above that $\mathscr{R}_1 A \cap \mathscr{R}_2 = \mathscr{R}_1 A \cap (\mathscr{R}_1 \cap \mathscr{R}_2) = \mathscr{R}_1 A \cap (\mathscr{R}_1 \cap \mathscr{R}_2)A = (\mathscr{R}_1 \cap \mathscr{R}_2)A$. $\square$

Corollary 4.2.10. *Let $\mathscr{R}_1, \mathscr{R}_2$ be von Neumann algebras acting on the Hilbert space $\mathscr{H}$. Let $\mathcal{S}$ be a family of operators in $\mathscr{R}_1 \cap \mathscr{R}_2$. Then $\langle \mathscr{R}_1 \mathcal{S} \rangle \cap \mathscr{R}_2 = \langle (\mathscr{R}_1 \cap \mathscr{R}_2)\mathcal{S} \rangle$.*

Proof. Let A, B be operators in $\mathcal{S}$. From Proposition 4.2.5, 4.2.10, we have that,

$$\begin{aligned}
\langle \mathscr{R}_1 \{A, B\} \rangle \cap \mathscr{R}_2 &= (\mathscr{R}_1 A + \mathscr{R}_1 B) \cap \mathscr{R}_2 = \mathscr{R}_1 \sqrt{A^*A + B^*B} \cap \mathscr{R}_2 \\
&= (\mathscr{R}_1 \cap \mathscr{R}_2)\sqrt{A^*A + B^*B} = (\mathscr{R}_1 \cap \mathscr{R}_2)A + (\mathscr{R}_1 \cap \mathscr{R}_2)B \\
&= \langle (\mathscr{R}_1 \cap \mathscr{R}_2)\{A, B\} \rangle
\end{aligned}$$

Thus $\langle \mathscr{R}_1 \mathcal{S} \rangle \cap \mathscr{R}_2 = \langle (\mathscr{R}_1 \cap \mathscr{R}_2) \mathcal{S} \rangle$. $\square$

The corollary below is in the same vein as Proposition 4.2.8. In effect, it says that every left ideal of a represented von Neumann algebra may be viewed as the intersection of a left ideal of the full algebra of bounded operators on the underlying Hilbert space with the von Neumann algebra.

Corollary 4.2.11. *Let $\mathscr{R}$ be a von Neumann algebra acting on the Hilbert space $\mathscr{H}$. Let $\mathscr{I}$ be a left ideal of $\mathscr{R}$. Then there is a left ideal $\mathscr{J}$ of $\mathcal{B}(\mathscr{H})$ such that $\mathscr{I} = \mathscr{J} \cap \mathscr{R}$.*

Proof. By choosing $\mathscr{R}_1 = \mathcal{B}(\mathscr{H}), \mathscr{R}_2 = \mathscr{R}$ and $\mathcal{S} = \mathscr{I}$ and using Corollary 4.2.10, we see that for $\mathscr{J} := \langle \mathcal{B}(\mathscr{H}) \mathscr{I} \rangle$, we have that, $\mathscr{I} = \langle \mathscr{R} \mathscr{I} \rangle = \langle (\mathcal{B}(\mathscr{H}) \cap \mathscr{R}) \mathscr{I} \rangle = \mathscr{J} \cap \mathscr{R}$ and $\mathscr{J}$ is a left ideal of $\mathcal{B}(\mathscr{H})$. $\square$

4.3 A Result on C^*-convexity

Lemma 4.3.1. *Let $\mathscr{R}$ be a von Neumann algebra acting on the Hilbert space $\mathscr{H}$, with identity I. If $T_1, \cdots, T_n$ are operators in $\mathscr{R}$ such that $T_1^* T_1 + \cdots + T_n^* T_n = I$, then there are operators $S_1, \cdots, S_{n-1}$ in $\mathscr{R}$ such that $T_i = S_i \sqrt{I - T_n^* T_n}, 1 \leq i \leq n-1$ and $S_1^* S_1 + \cdots S_{n-1}^* S_{n-1}$ is the range projection of $\sqrt{I - T_n^* T_n}$.*

Proof. As $T_1^* T_1 + \cdots + T_{n-1}^* T_{n-1} = I - T_n^* T_n$, we have that

$$T_i^* T_i \leq \sqrt{I - T_n^* T_n} \sqrt{I - T_n^* T_n}, 1 \leq i \leq n-1.$$

Since $\sqrt{I - T_n^*T_n}$ is self-adjoint, the orthogonal complement of the range of $\sqrt{I - T_n^*T_n}$ is equal to the kernel of $\sqrt{I - T_n^*T_n}$. From the proof of Theorem 3.2.1, we have for each $i \in \{1, 2, \cdots, n-1\}$, an operator S_i in $\mathscr{R}$ such that $T_i = S_i\sqrt{I - T_n^*T_n}$, and $S_ih = 0$ for any vector h in $\ker(\sqrt{I - T_n^*T_n})$. Note that,

$$I - T_n^*T_n = T_1^*T_1 + \cdots + T_{n-1}^*T_{n-1} = \sqrt{I - T_n^*T_n}(S_1^*S_1 + \cdots + S_{n-1}^*S_{n-1})\sqrt{I - T_n^*T_n}.$$

As a result, for every vector f in the range of $\sqrt{I - T_n^*T_n}$, we have $\langle f, f\rangle = \langle (S_1^*S_1 + \cdots + S_{n-1}^*S_{n-1})f, f\rangle$. In addition, for every vector in the kernel of $\sqrt{I - T_n^*T_n}$, we have $\langle (S_1^*S_1 + \cdots + S_{n-1}^*S_{n-1})f, f\rangle = 0$. Thus the operator $S_1^*S_1 + \cdots + S_{n-1}^*S_{n-1}$ must be the range projection of $\sqrt{I - T_n^*T_n}$. $\square$

Proposition 4.3.2. *Let $\mathscr{R}$ be a finite von Neumann algebra. Then a subset $\mathscr{S}$ of $\mathscr{R}$ is C^*-convex in $\mathscr{R}$ if and only if the C^*-segment $S(A_1, A_2)$ joining A_1 and A_2 lies in $\mathscr{S}$ for all A_1, A_2.*

Proof. If $\mathscr{S}$ is C^*-convex, $S(A_1, A_2)$ is clearly in $\mathscr{S}$ for any $A_1, A_2 \in \mathscr{S}$ as it consists of C^*-convex combinations of A_1 and A_2.

For the other direction, we inductively prove that for $(A_1, \cdots, A_n)$, an n-tuple with entries in $\mathscr{S}$ and $T_1, \cdots, T_n \in \mathscr{R}$ satisfying $T_1^*T_1 + \cdots T_n^*T_n = I$, the C^*-convex combination $T_1^*A_1T_1 + \cdots T_n^*A_nT_n$ is in $\mathscr{S}$. For $n = 1, 2$, the above is clearly true. Assuming that it is true for $n-1$, we will prove that it is also true for n.

As $\mathscr{R}$ is finite, by 6.9.10(ii) in [12], each of the T_i's has a unitary polar decomposition i.e. there are unitary operators U_i and positive operators P_i such that $T_i = U_iP_i, 1 \leq i \leq n$. The operators $A'_i := U_i^* A_i U_i$ are in $\mathscr{S}$ and $P_1^2 + \cdots + P_n^2 = I$. From Lemma 4.3.1, there are operators $S_1, \cdots, S_{n-1}$ as defined in 4.3.1 in $\mathscr{R}$ such that $P_i = S_i\sqrt{I - P_n^2}, 1 \leq i \leq n-1$ and $S_1^*S_1 + \cdots + S_{n-1}^*S_{n-1} = E$ where E is the range projection of $\sqrt{I - P_n^2}$. Let F denote the projection onto the kernel of $\sqrt{I - P_n^2}$. As $F = I - E$, clearly F is in $\mathscr{R}$. For $i \in \{1, \cdots, n-1\}$ as $\ker(P_i) \subseteq \ker(\sqrt{I - P_n^2})$, we have that $P_iF = FP_i = 0$ and as a result $FS_i = 0$. Define $S'_i := S_i + \frac{F}{\sqrt{n-1}}, 1 \leq i \leq n-1$. We see that $S_1'^*S'_1 + \cdots + S_{n-1}'^*S'_{n-1} = (S_1^*S_1 + \frac{F}{n-1}) + \cdots + (S_{n-1}^*S_{n-1} + \frac{F}{n-1}) = E + F = I$ and also $P_i = S'_i\sqrt{I - P_n^2}$. By the inductive hypothesis $A' := S_1'^*A'_1S_1 + \cdots S_{n-1}'^*A'_{n-1}S'_{n-1}$ is in $\mathscr{S}$. As $T_1^*A_1T_1 + \cdots T_n^*A_nT_n = \sqrt{I - P_n^2}A\sqrt{I - P_n^2} + P_nA'_nP_n$, being a C^*-convex combination of A' and A'_n, it must be in $\mathscr{S}$.

$\square$

Chapter 5

The Abelian Case

In this chapter, we give a characterization of diagonals of projections in $M_n(\mathfrak{A})$ where $\mathfrak{A}$ is an abelian C^*-algebra. The classical Gelfand-Neumark theorem tells us that $\mathfrak{A}$ is *-isomorphic to the algebra of complex-valued continuous functions on a locally compact Hausdorff space X with involution as complex conjugation ($\mathfrak{A} \simeq C(X)$). Thus we can think of $M_n(\mathfrak{A})$ as a $n \times n$ matrix with entries (f_{ij}) as complex-valued continuous functions over X. In [3], Grove and Pedersen give topological conditions that the space X must satisfy for one to be able to diagonalize self-adjoint operators in $M_n(C(X))$. Keeping this in mind, it may appear that the topology of the state space of $\mathfrak{A}$ (i.e. X) should play a role in a result about diagonals of projections but as we will see later in this chapter, that is not the case.

In our discussion below, unless stated otherwise, we will consider $M_n(\mathbb{C})$ as a normed space with norm for a matrix A given by $\|A\| := \sup_{i,j} |a_{ij}|$. Similarly, we consider the norm on $\mathbb{C}^n$ given by $\|v\| = \sup_i |v_i|$. The subsets of $M_n(\mathbb{C}), \mathbb{C}^n$ will inherit the above metric. Being finite-dimensional spaces, although all norms are equivalent, we make this particular choice as it makes our proofs simpler.

Definition 5.0.3. *Let $\tilde{x} = (x_1, x_2, \cdots, x_n)$ be a point in $\mathbb{R}^n$ and S_n denote the group of permutations of $\{1, 2, \cdots, n\}$. Let $\mathscr{K}_{\tilde{x}}$ be the closed convex hull of $\{(x_{\pi(1)}, x_{\pi(2)}, \cdots, x_{\pi(n)} : \pi \in S_n\}$. $\mathscr{K}_{\tilde{x}}$ is referred to as the permutation polytope generated by $\tilde{x}$*

Let $\varphi^{(n)}$ be the mapping that assigns to each self-adjoint $n \times n$ matrix $(a_{jk}) \in M_n(\mathbb{C})$ the vector $(a_{11}, a_{22}, \cdots, a_{nn})$ in $\mathbb{R}^n$. For $0 \leq m \leq n$, let $\mathscr{K}_m^{(n)}$ denote the permutation polytope of the point in $\mathbb{R}^n$ whose first m coordinates are 1 and whose last $n - m$ coordinates are 0. Let $\mathscr{P}_m^{(n)}$ be the set of rank m projection matrices in $M_n(\mathbb{C})$. Then clearly, $\varphi^{(n)}$ is a continuous map such that $\varphi^{(n)}(\mathscr{P}_m^{(n)}) = \mathscr{K}_m^{(n)}$. Below, we inductively construct a map $\psi_m^{(n)} : \mathscr{K}_m^{(n)} \to \mathscr{P}_m^{(n)}$, as described in [1], such that $\varphi^{(n)} \circ \psi_m^{(n)} = id_{\mathscr{K}_m^{(n)}}$ and the range of $\psi_m^{(n)}$ consists of matrices with real entries.

When n is clear from the context, we suppress it in the notation and write φ, ψ_m instead of $\varphi^{(n)}, \psi_m^{(n)}$ respectively.

5.1 The Construction of $\psi_m^{(n)}$

For $n \in \mathbb{N}$ and $m = 1$, define $\psi_1^{(n)} : \mathscr{K}_1^{(n)} \to \mathscr{P}_1^{(n)}$ by

$$\psi_1^{(n)}(a_1, a_2, \cdots, a_n) := \begin{pmatrix} a_1 & \sqrt{a_1 a_2} & \cdots & \cdots & \sqrt{a_1 a_n} \\ \sqrt{a_2 a_1} & a_2 & \cdots & \cdots & \cdots \\ \vdots & \vdots & \ddots & \sqrt{a_i a_j} & \vdots \\ \vdots & \vdots & \sqrt{a_j a_i} & \ddots & \vdots \\ \sqrt{a_n a_1} & \cdots & \cdots & \cdots & a_n \end{pmatrix}.$$

i.e. $(\psi_1^{(n)}(a_1, a_2, \cdots, a_n))_{ij} = \sqrt{a_i a_j}$. Its easy to see that the range of $\psi_1^{(n)}$ consists of projection matrices with real entries.

Suppose we have constructed $\psi_{m-1}^{(n)}$ for all $n \geq m-1$ such that it has range consisting of real entries.

Assume that $a_1 + a_2 + \cdots + a_n = m$. Let k be the smallest integer j for which $a_1 + a_2 + \cdots + a_j \geq m - 1$ and a be $m - 1 - \sum_{r=1}^{k-1} a_r$. By the inductive hypothesis, $E_1 := \psi_{m-1}^{(n)}(a_1, a_2, \cdots, a, 0, \cdots, 0)$ is an orthogonal projection.

Let F_1 be E_1 with the $k+1, k+1$ entry replaced by 1. From Lemma 1, we have that the (j, r)th entry of E_1 is 0 if any of j, r is strictly bigger than k.($\because$, atleast one of a_{jj} or a_{kk} would be 0 in that case.) Thus F_1 is an orthogonal projection.

Let $W_k(t)$ denote the unitary matrix which has $\sqrt{t}$ at the (k,k) and $(k+1,k+1)$ entries, $-\sqrt{1-t}$ and $\sqrt{1-t}$ at the $(k,k+1)$ and $(k+1,k)$ entries, respectively, 1 at all other diagonal entries, and 0 at all other off-diagonal entries. Below is a representation of what $W_k(t)$ looks like,

Let $p(k,t,F_1)$ be $W_k(t)F_1W_k(t)^*$. The matrix $p(k,t,F_1)$ has diagonal entries $a_1,\cdots,a_{k-1},ta+(1-t),(1-t)a+t,0,\cdots,0$. The only entries that are different for F_1 and $p(k,r,F_1)$ are the kth row and column and the $(k+1)$st row and column. The entries in the kth column are $a_{1k}\sqrt{t},\cdots,a_{k-1,k}\sqrt{t},ta+(1-t),(a-1)\sqrt{t(1-t)},0,\cdots,0$. The entries in the $(k+1)$st column are $a_{1k}\sqrt{1-t},\cdots,a_{k-1,k}\sqrt{1-t},(a-1)\sqrt{t(1-t)},(1-t)a+t,0,\cdots,0$. Using the fact that F_1 is self-adjoint, one may see what the kth and $(k+1)$st rows look like.

By our choice of k, we have that $a = m-1-\sum_{r=1}^{k-1} a_r \leq a_k \leq 1$. Note that if $a=1$, then $a_k = 1$. If $a \neq 1$, we choose $t_1 = \frac{1-a_k}{1-a}$ and for $a=1$, we choose $t_1 = 1$, so that $t_1 a + (1-t_1)\cdot 1 = a_k$, and

$$(1-t_1)a + t_1 = a + t_1(1-a) = m-1-(\sum_{r=1}^{k-1} a_r) + 1 - a_k = m - \sum_{r=1}^{k} a_k$$

Let $p(k,t_1,F_1)$ be F_2. Each entry in the matrix for F_2 is real.

The projection $p(k+1,t,F_2)$ has as its diagonal entries

$$a_1,a_2,\cdots,a_k,(\sum_{r=k+1}^{n} a_r)t,(\sum_{r=k+1}^{n} a_r)(1-t),0,\cdots,0.$$

As $\sum_{r=k+1}^{n} a_r \neq 0$, we choose $t_2 = \frac{a_{k+1}}{\sum_{r=k+1}^{n} a_r}$. Thus, the projection $p(k+1, t_2, F_2)(= F_3)$ has as its diagonal entries $a_1, \cdots, a_{k+1}, \sum_{r=k+2}^{n} a_r, 0, \cdots, 0$. We continue with this construction forming $p(k+2, t_3, F_3)$ next and so forth. If at the l^{th} step, we see that $\sum_{r=k+l}^{n} a_r = 0$, we must have $a_r = 0$ for $k+l \leq r \leq n$, and we terminate the construction at that step and define $\psi_m^{(n)}(a_1, a_2, \cdots, a_n) = F_{l+1}$. Otherwise using $t_{l+1} = \frac{a_{k+l}}{\sum_{r=k+l}^{n} a_r}$, we continue the construction until we consider $p(n-1, t, F_{n-k})$. In this final step, we choose $t_{n-k} = \frac{a_{n-1}}{a_{n-1}+a_n}$, we let F_{n-k+1} be the orthogonal projection matrix $p(n-1, t_{n-k}, F_{n-k})$. The diagonal entries of F_{n-k+1} are $a_1, a_2, \cdots, a_n$, and all entries are real. We define $\psi_m^{(n)}(a_1, a_2, \cdots, a_n) := F_{n-k+1}$.

5.2 The Main Results

Proposition 5.2.1. *In the construction above, for $n \in \mathbb{N}$ and $1 \leq m \leq n$, the map $\psi_m^{(n)} : \mathscr{K}_m^{(n)} \to \mathscr{P}_m^{(n)}$ is continuous and has range in the set of rank m projection matrices with real entries.*

Proof. We prove this result inductively. For $m = 1$ and $n \in \mathbb{N}$, the continuity of $\psi_1^{(n)}$ is fairly straightforward from the expression in the construction above.

Let $\widetilde{a} = (a_1, a_2, \cdots, a_n) \in \mathscr{K}_m^{(n)}$. Let $\varepsilon > 0$ be given. In order to prove that $\psi_m^{(n)}$ is continuous at $\widetilde{a}$, we need to prove the existence of $\delta(\varepsilon) > 0$ such that $\|\psi_m^{(n)}(\widetilde{x}) - \psi_m^{(n)}(\widetilde{a})\| < \varepsilon$ whenever $\|\widetilde{x} - \widetilde{a}\| < \delta(\varepsilon)$ and $\widetilde{x} \in \mathscr{K}_m^{(n)}$. We will do so by splitting $\mathscr{K}_m^{(n)}$ into three regions and proving continuity of $\psi_m^{(n)}$ at points of each of these regions.

Case I : For any $r \in \{1, 2, \cdots, n-1\}$, $\sum_{i=1}^{r} a_i \neq m-1$ and $\sum_{i=r+1}^{n} a_i \neq 0$.

Let $k(= k(\widetilde{a}))$ which depends on the point $\widetilde{a}$ in $\mathscr{K}_m$ under consideration, be as chosen previously. Let $\delta_1 := \inf\{|m-1-\sum_{i=1}^{r} a_i|, \sum_{i=r+1}^{n} a_r\}_{r=1}^{n-1}$. If $\|\widetilde{x}-\widetilde{a}\| < \frac{\delta_1}{n}$, then $\widetilde{x}$ also satisfies the requirements for Case I and $k(\widetilde{x}) = k(\widetilde{a})$ i.e. the choice of k for $\widetilde{x}$ is the same as that for $\widetilde{a}$. If $\|\widetilde{x}-\widetilde{a}\| < \frac{\delta_2}{n}$, then $\|(x_1, x_2, \cdots, m-1-\sum_{r=1}^{k-1} x_r, 0, \cdots, 0) - (a_1, a_2, \cdots, m-1-\sum_{r=1}^{k-1} a_r, 0, \cdots, 0)\| < \delta_2$. By our inductive hypothesis, as $\psi_{m-1}^{(n)}$ is continuous, we can choose $\delta_2 (< \delta_1)$ sufficiently small to ensure $E_1(\widetilde{x}) := \psi_{m-1}^{(n)}(x_1, x_2, \cdots, m-1-\sum_{r=1}^{k-1} x_r, 0, \cdots, 0)$ is a continuous function in the $\frac{\delta_2}{n}$-neighborhood of $\widetilde{a}$ in $\mathscr{K}_m^{(n)}$.

We can see that the choice of $t_1(= t_1(\widetilde{x}) := \frac{1-x_k}{1-(m-1-\sum_{r=1}^{k-1} x_r)})$ is continuous in terms of $\widetilde{x}$, as are the other $t_i(\widetilde{x}) := \frac{x_{k+i-1}}{\sum_{r=k+i-1}^{n} x_r}$. At the end of the construction, we have that, for $\widetilde{x}$ in a small enough neighbourhood of $\widetilde{a}$ in $\mathscr{K}_m^{(n)}$, the entries of the constructed matrix $\psi_m^{(n)}(\widetilde{x})$ are continuous functions involving entries of $E_1(\widetilde{x}), t_i(\widetilde{x})$'s, $\widetilde{x}$, and hence a continuous function in $\widetilde{x}$. Thus $\psi_m^{(n)}$ is continuous at every point of the region described in Case I.

Case II : For some $r \in \{1, 2, \cdots, n-1\}$, $\sum_{i=1}^{r} a_i = m-1$ but $\sum_{i=r+1}^{n} a_i \neq 0$.

This case is more delicate than Case I, as the value of k chosen in the construction, may vary in a neighborhood of $\widetilde{a}$ and t_1 may not be a continuous function in any neighbourhood of $\widetilde{a}$. For this case, k is the smallest number such that $\sum_{i=1}^{k} a_i = m-1$. Hence $a_k \neq 0$. Let $W_{\widetilde{a}}$ be a small enough neighbourhood of $\widetilde{a}$ so that for $\widetilde{x} \in W_{\widetilde{a}}$, we have $\|\widetilde{x} - \widetilde{a}\| < \frac{a_k}{n}$. Then $k(\widetilde{x}) \geq k(\widetilde{a})$.

Subcase (i): $\underline{k(\widetilde{x}) = k(\widetilde{a})(= k)}$

If $a_k \neq 1$, by making $W_{\widetilde{a}}$ sufficiently smaller (and reusing the notation), we can ensure that $x_k \neq 1$ and by mimicking the proof used in Case I, for $\widetilde{x} \in W_{\widetilde{a}}$, we have $\|\psi_m^{(n)}(\widetilde{x}) - \psi_m^{(n)}(\widetilde{a})\| < \varepsilon$.

If $a_k = 1$, then $a = m - 1 - \sum_{i=1}^{k-1} a_i = a_k = 1$. By Remark 3.1.2, all the non-diagonal elements in the kth row and kth column of $F_1(\widetilde{a}), F_2(\widetilde{a})$ are zero. For a given $\varepsilon > 0$, we can choose a sufficiently small neighbourhood $W_{\widetilde{a}}$ of $\widetilde{a}$, so that for $\widetilde{x} \in W_{\widetilde{a}}$, we have $\|F_2(\widetilde{x}) - F_2(\widetilde{a})\| < \varepsilon$. Here we circumvent around the fact that t_1 may not be continuous by noting that if x_k is sufficiently close to 1 (say, $|1 - x_k| \leq \delta$), by Lemma 3.1.1, all the non-diagonal elements in the kth row and kth column of $F_1(\widetilde{x}), F_2(\widetilde{x})$ are sufficiently small (i.e. less than $\sqrt{\delta}$). Since t_i is continuous for $2 \leq i \leq n-k$, the proof works as in the Case I.

Subcase (ii): $\underline{k' := k(\widetilde{x}) > k(\widetilde{a})}$

As $\sum_{i=1}^{k'-1} x_i < m - 1 \leq \sum_{i=1}^{k'} x_i$, for a sufficiently small neighbourhood $W_{\widetilde{a}}$ (say, $\{x \in \mathscr{K} : \|\widetilde{x} - \widetilde{a}\| \leq \frac{\delta}{2n^2}\}$), $a_{k+1}, \cdots, a_{k'-1}$ must each have size smaller than $\frac{\delta}{n}$. Thus $x_{k+1}, \cdots, x_{k'-1}$ must each have size smaller than $\frac{\delta}{n} + \frac{\delta}{n^2} \leq 2\frac{\delta}{n}$. After the first $k' - k + 1$ steps of the construction for $\widetilde{a}$ we reach the same level as $F_1(\widetilde{x})$. Using Lemma 3.1.1, we have that $(F_{k'-k+1}(\widetilde{a}))_{ij} \leq \sqrt{\frac{\delta}{n}}$ and $(F_1(\widetilde{x}))_{ij} \leq \sqrt{2\frac{\delta}{n}}$ for i or j in $\{k+1, \cdots, k'-1\}$. The entry at $(k'+1, k'+1)$ for $F_1(\widetilde{x})$ is 1 and for $F_{k'-k+1}(\widetilde{a})$ it is $1 - \sum_{i=k+1}^{k'} a_i$. The vectors $(a_1, \cdots, a_{k'}, 1 - \sum_{i=k+1}^{k'} a_i)$ and $(x_1, \cdots, m-1-\sum_{i=1}^{k'-1} x_i, 1, 0, \cdots, 0)$ are sufficiently close (less than δ apart) that from this stage onwards, as the operations involving the t_is are continuous, by making the neighbourhood $W_{\widetilde{a}}$ smaller if need be, we get that $\|\psi_m^{(n)}(\widetilde{x}) - \psi_m^{(n)}(\widetilde{a})\| \leq \varepsilon$ for a given $\varepsilon > 0$, for $\widetilde{x} \in W_{\widetilde{a}}$.

Since there are finitely many possibilities for k', we take $\delta(\varepsilon)$ to be the minimum of all the $\delta(\varepsilon)_{k'}$'s. Thus $\psi_m^{(n)}$ is continuous at points in the above region.

<u>Case III :</u> For some $r \in \{1, 2, \cdots, n-1\}$, we have $\sum_{i=r+1}^{n} a_i = 0$.

Our construction is terminated as soon as we reach a_r. But for $\widetilde{x}$ in a neighborhood of $\widetilde{a}$, the construction might as well continue. Let $j \geq r+1$. Then $a_j = 0$ and $(\psi_m^{(n)}(\widetilde{a}))_{ij} = 0$ for $i \in \{1, 2, \cdots, n\}$ as using Lemma 3.1.1, we get $0 \leq |(\psi_m^{(n)}(\widetilde{a}))_{ij}| \leq \sqrt{a_i} \cdot \sqrt{a_j} = 0$. If $\|\widetilde{x} - \widetilde{a}\| < \delta(\varepsilon)$, then $|b_j - a_j| = |b_j| < \delta(\epsilon)$. By a suitable choice, one can ensure $\delta(\varepsilon) < \varepsilon^2$ and thus from Lemma 3.1.1, we have $|(\psi_m^{(n)}(\widetilde{x}))_{ij} - (\psi_m^{(n)}(\widetilde{a}))_{ij}| = |(\psi_m^{(n)}(\widetilde{x}))_{ij}| \leq \sqrt{x_i} \cdot \sqrt{x_j} < \varepsilon$. For $j \leq r$, one may mimic the relevant parts of the proofs in Cases I and II for the choice of $\delta(\varepsilon)$, to prove that the matrices $\psi_m^{(n)}(\widetilde{a})$ and $\psi_m^{(n)}(\widetilde{x})$ are entry-wise at the most ε-distance apart.

Now that all the possible cases have been taken care of, the proof is done.

$\square$

Definition 5.2.2. *Let X be a locally compact Hausdorff space. We denote the set of continuous functions from X to a subset B of $\mathbb{C}^n$ or $\mathbb{R}^n$ by $C(X;B)$ and the space of functions in $C(X;\mathbb{C}^n)$ vanishing at infinity, by $C_0(X;\mathfrak{A})$. $C_0(X;\mathfrak{A})$ naturally has the structure of a C^*-algebra with pointwise addition and multiplication, and complex conjugation as involution.*

Lemma 5.2.3. *Let $\mathfrak{A}$ be a unital abelian C^*-algebra such that $\mathfrak{A} \simeq C(X;\mathbb{C})$, for some compact Hausdorff space X, with identity $1_{\mathfrak{A}}$. Let φ be the mapping that assigns to each self-adjoint $n \times n$ matrix $(a_{jk}(\cdot)) \in M_n(\mathfrak{A})$ the vector-valued function $(a_{11}(\cdot), a_{22}(\cdot), \cdots, a_{nn}(\cdot))$ from X to $\mathbb{R}^n$ (an element of $(\mathfrak{A}^{\mathfrak{sa}})^n \subset \mathfrak{A}^n$).*
If $a_1, a_2, \cdots, a_n \in \mathfrak{A}$ such that $0 \le a_i \le 1_{\mathfrak{A}}$, for $i \in \{1, \cdots, n\}$ and $\sum_{i=1}^n a_i = m \cdot 1_{\mathfrak{A}}$, then there exists a projection in $M_n(\mathfrak{A})$ with $(a_1, a_2, \cdots, a_n)$ as diagonal.
In other words, $C(X;\mathscr{K}_m) \subseteq \varphi(\mathscr{P}(M_n(\mathfrak{A})))$, where $\mathscr{P}(M_n(\mathfrak{A}))$ is the set of projections in $M_n(\mathfrak{A})$.

Proof. Let $F \in C(X;\mathscr{K}_m)$. Consider the function ψ_m whose existence was proven in Proposition 1. Let $G := \psi_m \circ F \in \mathscr{P}(M_n(\mathfrak{A}))$. Then $\varphi \circ G = (\varphi \circ \psi_m) \circ F = F$.

$$\begin{tikzcd}
 & \mathscr{P}(M_n(\mathfrak{A})) \arrow[d, "\varphi"'] \\
X \arrow[ur, "G"] \arrow[r, "F"'] & \mathscr{K}_m \arrow[u, bend right, "\psi_m"']
\end{tikzcd}$$

Thus $F \in \phi(\mathscr{P}(M_n(\mathfrak{A}))$. This proves that $C(X;\mathscr{K}_m)$ is contained in $\varphi(\mathscr{P}(M_n(\mathfrak{A})))$.

$\square$

Now we are in a position to characterize the diagonals of projections in $M_n(\mathfrak{A})$ where $\mathfrak{A}$ is any abelian C^*-algebra.

Theorem 5.2.4. *Let $\mathfrak{A}$ be an abelian C^*-algebra. Let $a_1, a_2, \cdots, a_n \in \mathfrak{A}$. Then there exists a projection in $M_n(\mathfrak{A})$ with diagonal $(a_1, a_2, \cdots, a_n)$ if and only if there exist mutually orthogonal projections $p_i \in \mathfrak{A}, i \in \{1, 2, \cdots, n\}$ such that $0 \le a_j \le \sum_{i=1}^n p_i$, $j \in \{1, 2, \cdots, n\}$ and $\sum_{i=1}^n a_i = \sum_{i=1}^n i \cdot p_i$.*

Proof. From the Gelfand-Neumark theorem, we know that $\mathfrak{A} \simeq C_0(X; \mathbb{C})$ for some locally compact Hausdorff space X.

($\Rightarrow$) Let P be the projection in $M_n(\mathfrak{A}) \simeq C_0(X; M_n(\mathbb{C}))$ with diagonal $(a_1, a_2, \cdots, a_n)$. We note that $s := \sum_{i=1}^n a_i$ is a continuous function on X and only attains integral values between 0 and n (both included). This is because, for $x \in X$, $s(x)$ is the trace of the projection $P(x)$. As $s^{-1}(\{i\}), i \in \{0, 1, \cdots, n\}$ are clopen sets of X, the projections $p_i = \chi_{s^{-1}(\{i\})}$ belong to $\mathfrak{A}$ (where χ_A denotes the characteristic function of the set A) and are such that $s = \sum_{i=1}^n i \cdot p_i$. Also the diagonal elements of $P(x)$ must lie between 0 and 1 for each x. Thus $0 \le a_j \le \sum_{i=1}^n p_i$ for $j \in \{1, 2, \cdots, n\}$. This proves that the conditions described are necessary.

($\Leftarrow$) Consider the unital abelian C^*-algebra $p_j \mathfrak{A} p_j = p_j \mathfrak{A}$ with unit p_j. As the p_i's are mutually orthogonal, $\sum_{i=1}^n p_j a_i p_j = p_j(\sum_{i=1}^n a_i) p_j = p_j(\sum_{i=1}^n i \cdot p_i) p_j = j \cdot p_j$. Now, as $0 \le p_j a_i p_j \le p_j(\sum_{k=1}^n p_k) p_j = p_j$, by appealing to Lemma 2, we observe that there is a projection P_j in $M_n(p_j\mathfrak{A}p_j)$ with diagonal $(p_ja_1p_j, p_ja_2p_j, \cdots, p_ja_np_j)$. $\sum_{i=1}^n p_iP_ip_i$ is a projection in $M_n(\mathfrak{A})$ satisfying the given conditions.

$\square$

Chapter 6

The Non-commutative Case

6.1 Diagonals of Projections in $M_2(M_n(\mathbb{C}))$

Let $M_n(\mathbb{C})$ denote the set of $n \times n$ complex matrices. For a Hermitian matrix A in $M_n(\mathbb{C})$, the vector in $\mathbb{R}^n$ whose entries are the eigenvalues of A (counted with multiplicity) arranged in decreasing order, is called the spectral distribution of A. We denote it by $m(A)$.

Proposition 6.1.1. *Let A_1, A_2 be Hermitian matrices in $M_{k_1}(\mathbb{C}), M_{k_2}(\mathbb{C})$ respectively, with eigenvalues in the open interval $(0,1)$. There is an orthogonal projection P in $M_{k_1+k_2}(\mathbb{C})$ such that A_1 and A_2 form the diagonal blocks of P if and only if $k_1 = k_2 (=: n)$ and $m(A_1) = m(I_n - A_2)$.*

Proof. Let P be a Hermitian matrix with diagonal blocks A_1, A_2 in block matrix form as below,

$$\begin{pmatrix} A_1 & B^* \\ B & A_2 \end{pmatrix}$$

where B is a $k_2 \times k_1$ matrix.

The matrix P is an orthogonal projection iff there exists a B such that $P^2 = P$. More explicitly, if and only if for some B

$$\begin{pmatrix} A_1^2 + B^*B & A_1B^* + B^*A_2 \\ BA_1 + A_2B & A_2^2 + BB^* \end{pmatrix} = \begin{pmatrix} A_1 & B^* \\ B & A_2 \end{pmatrix}.$$

The above expression essentially gives us three equations.

$$B^*B = A_1 - A_1^2,$$

$$BB^* = A_2 - A_2^2,$$

$$BA_1 = (I_{k_2} - A_2)B.$$

First, we prove the necessity of the given conditions in the statement of the proposition. As the matrices A_1, A_2 have eigenvalues in the open interval $(0, 1)$, by the spectral mapping theorem we observe that $A_1 - A_1^2, A_2 - A_2^2$ must have eigenvalues in $(0, \frac{1}{4}]$ and hence they are invertible. From the preceding set of equations, we have that BB^* and B^*B are invertible. As in general, $\text{rank}(BB^*) = \text{rank}(B^*B)$, we conclude that $k_1 = k_2$ and B must be an invertible matrix.

The third equation may now be rewritten as $BA_1B^{-1} = I_n - A_2$ where $n := k_1 (= k_2)$. Since similar matrices have the same set of eigenvalues counted with multiplicity, we note that $m(A_1) = m(BA_1B^{-1}) = m(I_n - A_2)$.

Now we prove the sufficiency of the conditions. Let $k_1 = k_2 = n$ and $m(A_1) = m(I_n - A_2)$. There is a unitary matrix U in $M_n(\mathbb{C})$ such that $UA_1U^* = I_n - A_2$. We define $B := U\sqrt{A_1 - A_1^2}$. We just need to check that the three equations above are satisfied by our choice of B. As $U^*U = UU^* = I_n$,

$B^*B = \sqrt{A_1 - A_1^2}U^*U\sqrt{A_1 - A_1^2} = A_1 - A_1^2,$

$BB^* = U(A_1 - A_1^2)U^* = UD_1U^* - (UD_1U^*)^2 = (I_n - A_2) - (I_n - A_2)^2 = A_2 - A_2^2,$

$BA_1B^{-1} = U(A_1 - A_1^2)^{1/2}A_1(A_1 - A_1^2)^{-1/2}U^* = UA_1U^* = I_n - A_2.$

□

Let $\mathscr{K}_n$ denote the compact convex subset of $\mathbb{R}^n$ consisting of vectors $\widetilde{v} := (\lambda_1, \cdots, \lambda_n)$ such that $1 \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$. The sum of the entries of $\widetilde{v}$ is called the *trace* of $\widetilde{v}$ and denoted by $\tau(\widetilde{v})$. As will be clearer later, we deliberately use the same notation for the trace of a matrix. In fact, for a Hermitian matrix $A \in M_n(\mathbb{C}), \tau(A) :=$ trace of $A = \tau(m(A))$.

We denote the vector $(1, \cdots, 1, 0, \cdots, 0)$ in $\mathscr{K}_n$ with trace k by $\widetilde{e}_k$ (n will be clear from the context). We call the vector obtained by truncation after removing the entries $1, 0$ from $\widetilde{v}$ in $\mathscr{K}_n$ as the *nucleus* of $\widetilde{v}$ and denote it by $\widetilde{v}^\circ$. Note that the nucleus of $\widetilde{v}$ has entries strictly between 0 and 1 arranged in decreasing order and is empty iff $\widetilde{v} = \widetilde{e}_k$ for some non-negative integer k.

Lemma 6.1.2. *Let A_1, A_2 be positive contractions in $M_n(\mathbb{C})$ such that $m(A_1) = m(I_n - A_2)$. Then $(m(A_1), m(A_2))$ is in the convex hull of $\{(\widetilde{e}_k, \widetilde{e}_{n-k}) : 0 \leq k \leq n, k \in \mathbb{Z}\} \subset \mathscr{K}_n \times \mathscr{K}_n$.*

Proof. Let $m(A_1) = (\mu_1, \cdots, \mu_n)$. As $m(A_1) = m(I_n - A_2)$, we have that $m(A_2) = m(I_n - A_1) = (1 - \mu_n, \cdots, 1 - \mu_1)$. We may write $m(A_1)$ as a convex combination of $\widetilde{e}_k$'s as follows :

$$m(A_1) = (\mu_1, \cdots, \mu_n) = (1 - \mu_1)\widetilde{e}_0 + (\mu_1 - \mu_2)\widetilde{e}_1 + \cdots + (\mu_{n-1} - \mu_n)\widetilde{e}_{n-1} + \mu_n\widetilde{e}_n.$$

It follows that,

$(1 - \mu_1, \cdots, 1 - \mu_n) = \widetilde{e}_n - (\mu_1, \cdots, \mu_n) = (1 - \mu_1)(\widetilde{e}_n - \widetilde{e}_0) + (\mu_1 - \mu_2)(\widetilde{e}_n - \widetilde{e}_1) + \cdots + (\mu_{n-1} - \mu_n)(\widetilde{e}_n - \widetilde{e}_{n-1}) + \mu_n(\widetilde{e}_n - \widetilde{e}_n)$

$\Rightarrow (1 - \mu_n, \cdots, 1 - \mu_1) = (1 - \mu_1)\widetilde{e}_n + (\mu_1 - \mu_2)\widetilde{e}_{n-1} + \cdots + (\mu_{n-1} - \mu_n)\widetilde{e}_1 + \mu_n\widetilde{e}_0.$

Thus $(m(A_1), m(A_2)) = (1 - \mu_1) \cdot (\widetilde{e}_0, \widetilde{e}_n) + \cdots + (\mu_i - \mu_{i-1}) \cdot (\widetilde{e}_i, \widetilde{e}_{i-1}) + \cdots + \mu_n \cdot (\widetilde{e}_n, \widetilde{e}_0)$. □

Remark 6.1.3. *If $(\widetilde{v}, \widetilde{w}) = t_0(\widetilde{e}_0, \widetilde{e}_n) + t_1(\widetilde{e}_1, \widetilde{e}_{n-1}) + \cdots + t_n(\widetilde{e}_n, \widetilde{e}_0) \in \mathscr{K}_n \times \mathscr{K}_n$ where $0 < t_0, t_n < 1$, then $\widetilde{v} = \widetilde{v}^\circ$ and $\widetilde{w} = \widetilde{w}^\circ$. In this case, we may note a partial converse to Lemma 6.1.2 using Proposition 6.1.1, that there is a orthogonal projection with diagonal blocks $\mathrm{diag}(\widetilde{v}), \mathrm{diag}(\widetilde{w})$.*

Remark 6.1.4. *Let A_1, A_2 be the diagonal blocks of an orthogonal projection given* *as follows :*

$$\begin{pmatrix} A_1 & B^* \\ B & A_2 \end{pmatrix}$$

From the computations in the proof of Proposition 6.1.1, we have that $A_1 - A_1^2 = B^*B, A_2 - A_2^2 = BB^*$. *Since* $B^*B = 0$ *if and only if* $BB^* = 0$, *we see that* A_1 *is an orthogonal projection if and only if* A_2 *is an orthogonal projection.*

For a positive contraction $A \in M_n(\mathbb{C})$, *the nucleus of* $m(A)$ *is empty if and only if* A *is a projection. Thus, we may reformulate the above observation in the following manner : if* A_1, A_2 *are the diagonal blocks of an orthogonal projection, then the nucleus of* $m(A_1)$ *is empty if and only if the nucleus of* $m(A_2)$ *is empty.*

Lemma 6.1.5. *Let* A_1, A_2 *be positive contractions in* $M_{k_1}(\mathbb{C}), M_{k_2}(\mathbb{C})$ *respectively. Then* A_1, A_2 *form the principal diagonal blocks of some orthogonal projection in* $M_{k_1+k_2}(\mathbb{C})$ *if and only if diag*$(m(A_1)^\circ)$, *diag*$(m(A_2)^\circ)$ *form the diagonal blocks of some orthogonal projection matrix or the nucleus of both* $m(A_1), m(A_2)$ *are empty.*

Proof. As A_1, A_2 are Hermitian, for $i = 1, 2$, there exist unitary matrices U_i in $M_{k_i \times k_i}(\mathbb{C})$ such that $U_i A_i U_i^*$ are diagonal matrices with diagonal vector given by $m(A_i)$. Let U be the unitary matrix in $M_{k_1+k_2}(\mathbb{C})$ given by,

$$\begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix}$$

If P is an orthogonal projection with diagonal blocks A_1, A_2, then UPU^* is an orthogonal projection with diagonal blocks $\mathrm{diag}(m(A_1)), \mathrm{diag}(m(A_2))$. Conversely, if Q is an orthogonal projection with diagonal blocks $\mathrm{diag}(m(A_1)), \mathrm{diag}(m(A_2))$, then U^*QU is an orthogonal projection with diagonal blocks A_1, A_2. This shows that A_1, A_2 form the principal diagonal blocks of some orthogonal projection in $M_{k_1+k_2}(\mathbb{C})$ if and only if $\mathrm{diag}(m(A_1)), \mathrm{diag}(m(A_2))$ form the diagonal blocks of some orthogonal projection matrix in $M_{k_1+k_2}(\mathbb{C})$.

We recall that for $i = 1, 2$, $m(A_i)$ is of the form $(1, \cdots, 1, m(A_i)^\circ, 0, \cdots, 0)$. From Remark 6.1.5 and Remark 6.1.6, it follows that if $\mathrm{diag}(m(A_1)), \mathrm{diag}(m(A_2))$ form the diagonal blocks of some orthogonal projection matrix in $M_{k_1+k_2}(\mathbb{C})$, then either the nucleus of both $m(A_1), m(A_2)$ are empty or $\mathrm{diag}(m(A_1)^\circ), \mathrm{diag}(m(A_2)^\circ)$ form the diagonal blocks of some orthogonal projection matrix.

For the converse, if the nucleus of both $m(A_1), m(A_2)$ are empty i.e. A_1, A_2 are projections, then the required orthogonal projection is given by :

$$\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$$

If not, and if we have that $\mathrm{diag}(m(A_1)^\circ), \mathrm{diag}(m(A_2)^\circ)$ form the diagonal blocks of some orthogonal projection P, we may form an orthogonal projection with diagonal blocks $\mathrm{diag}(m(A_1)), \mathrm{diag}(m(A_2))$ by extending P and filling in 0's in the rows and the columns corresponding to the diagonal entries $0, 1$. $\square$

Theorem 6.1.6. *Let A_1, A_2 be two Hermitian matrices in $M_{k_1}(\mathbb{C}), M_{k_2}(\mathbb{C})$ respec-*

tively. Then A_1, A_2 appear as the principal diagonal blocks of a rank n orthogonal projection P in $M_{k_1+k_2}(\mathbb{C})$ if and only if $(m(A_1), m(A_2))$ is in the convex hull of $\{(\widetilde{e}_r, \widetilde{e}_s) : 0 \le r \le k_1, 0 \le s \le k_2, r+s=n\} \subset \mathscr{K}_{k_1} \times \mathscr{K}_{k_2}$.

Proof. Let P be an orthogonal projection with principal diagonal blocks A_1, A_2. If the nucleus of both $m(A_1), m(A_2)$ are empty, then $(m(A_1), m(A_2)) = (\widetilde{e}_r, \widetilde{e}_s)$ and $n = \operatorname{rank}(P) = \tau(P) = \tau(A_1) + \tau(A_2) = \tau(\widetilde{e}_r) + \tau(\widetilde{e}_s) = r + s$.

Next we may assume that the nucleus of neither of neither $m(A_1)$ nor $m(A_2)$ are empty. Let $m(A_1) = (1, \cdots, 1, \lambda_1, \cdots, \lambda_p, 0, \cdots, 0)$ and $m(A_2) = (1, \cdots, 1, \mu_1, \cdots, \mu_q, 0, \cdots, 0)$ where $m(A_1)^\circ = (\lambda_1, \cdots, \lambda_p), m(A_2)^\circ = (\mu_1, \cdots, \mu_q)$. From Lemma 6.1.5, $\operatorname{diag}(m(A_1)^\circ)$, $\operatorname{diag}(m(A_2)^\circ)$ must also be the diagonal blocks of an orthogonal projection. Then Proposition 6.1.1 tells us that $p = q$ and $\mu_i = 1 - \lambda_{p-i+1}, 1 \le i \le p$. By Lemma 7.1.2, $(m(A_1)^\circ, m(A_2)^\circ)$ is in the convex hull of $\{(\widetilde{e}_k, \widetilde{e}_{p-k}) : 0 \le k \le p\} \subset \mathscr{K}_p \times \mathscr{K}_p$. Thus $(m(A_1), m(A_2))$ is in the convex hull of $\{(\widetilde{e}_r, \widetilde{e}_s) : 0 \le r \le k_1, 0 \le s \le k_2, r+s=n\} \subset \mathscr{K}_{k_1} \times \mathscr{K}_{k_2}$.

We prove the sufficiency of the condition below. Without loss of generality, we may assume that $k_1 \ge k_2$. Let $(\widetilde{v}, \widetilde{w})$ belong to the convex hull of $\{(\widetilde{e}_r, \widetilde{e}_s) : 0 \le r \le k_1, 0 \le s \le k_2, r+s=n\}$. We may write it as a convex combination in the following manner $t_1(\widetilde{e}_{r_1}, \widetilde{e}_{n-r_1}) + t_2(\widetilde{e}_{r_2}, \widetilde{e}_{n-r_2}) + \cdots + t_m(\widetilde{e}_{r_m}, \widetilde{e}_{n-r_m})$ such that $0 \le r_1 \le \cdots \le r_m \le \min(k_1, n)$ and $0 < t_i < 1, i \in \{1, 2, \cdots, m\}$. In this case we note that $\widetilde{v}^\circ$ is in $\mathscr{K}_{r_m - r_1}$, and $\widetilde{w}^\circ \in \mathscr{K}_{(n-r_1)-(n-r_m)} = \mathscr{K}_{r_m - r_1}$. In fact $(\widetilde{v}^\circ, \widetilde{w}^\circ) = t_1(\widetilde{e}_0, \widetilde{e}_{r_m - r_1}) + t_2(\widetilde{e}_{r_2 - r_1}, \widetilde{e}_{r_m - r_2}) + \cdots + t_m(\widetilde{e}_{r_m}, \widetilde{e}_0) \in \mathscr{K}_{r_m - r_1} \times \mathscr{K}_{r_m - r_1}$. As $0 <$

$t_1, t_m < 1$, by Remark 6.1.3, there is an orthogonal projection with diagonal blocks $\operatorname{diag}(\widetilde{v}^\circ), \operatorname{diag}(\widetilde{w}^\circ)$. Thus from Lemma 6.1.5, we conclude that there is an orthogonal projection with diagonal blocks A_1, A_2 such that $m(A_1) = \widetilde{v}, m(A_2) = \widetilde{w}$.

□

6.2 Operator Inequalities

Let $\mathscr{R}$ be a von Neumann algebra and A in $M_n(\mathscr{R})$ be a self-adjoint element. By Kadison's diagonalization theorem ([9]), there exists a unitary U in $M_n(\mathscr{R})$ such that UAU^* is in diagonal form $D := \operatorname{diag}(D_1, D_2, \cdots, D_n)$ where D_i's are self-adjoint elements in $\mathscr{R}$.

Proposition 6.2.1. *Let the diagonal of A in $M_n(\mathscr{R})$ be $(A_{11}, \cdots, A_{nn})$. Then each of the diagonal elements $A_{ii}, 1 \leq i \leq n$ is in the C^*-polytope generated by $(D_1, \cdots, D_n)$.*

Proof. From the choice of U, we see that $A = U^* \operatorname{diag}(D_1, D_2, \cdots, D_n)U$. Thus $A_{jj} = \sum_{i=1}^n U_{ij}^* D_i U_{ij}$ and as U is unitary, $U^*U = I$ which yields $\sum_{i=1}^n U_{ij}^* U_{ij} = I \in \mathscr{R}, \forall j, 1 \leq j \leq n$. This proves that A_{jj} is in the C^*-polytope generated by the D_i's.

□

Lemma 6.2.2. *Let $A_1, \cdots, A_n$ be self-adjoint elements in $\mathscr{R}$ with spectrum in the interval $[a, b] \subset \mathbb{R}$. Then any C^*-convex combination of the A_i's also has spectrum in $[a, b]$.*

Proof. Let $U_1, \cdots, U_n$ be elements in $\mathscr{R}$ such that $\sum_{i=1}^n U_i^* U_i = I$. As the A_i's have spectrum in $[a, b]$, we have that $aI \leq A_i \leq bI$ for $i \in \{1, 2, \cdots, n\}$. Thus $aU_i^* U_i \leq U_i^* A_i U_i \leq bU_i^* U_i$ for all i and adding the inequalities we get that $aI \leq \sum_{i=1}^n U_i^* A_i U_i \leq bI$. From this, we conclude that $\sum_{i=1}^n U_i^* A_i U_i$ has spectrum in $[a, b]$. $\square$

Theorem 6.2.3. *Let $\mathscr{R}$ be a von Neumann algebra which contains a copy of $M_m(\mathbb{C})$ for all m in $\mathbb{N}$. Let $\Phi : M_n(\mathscr{R}) \to M_n(\mathscr{R})$ be the diagonal mapping which takes an operator in $M_n(\mathscr{R})$ to its diagonal by changing the off-diagonal entries to 0. Let f be a continuous function on the interval $[a, b] \subset \mathbb{R}$. Then f is operator-convex if and only if $f(\Phi(A)) \leq \Phi(f(A))$ for all self-adjoint operators A in $M_n(\mathscr{R})$ with spectrum in $[a, b]$.*

Proof. Let A be a self-adjoint element in $M_n(\mathscr{R}), n \geq 2$ wih spectrum in I. By Kadison's diagonalization theorem ([9]), there is a unitary operator U in $M_n(\mathscr{R})$ such that $D := UAU^*$ is diagonal. Let $D = \text{diag}(D_1, \cdots, D_n)$ where D_is are self-adjoint elements in $\mathscr{R}$. As the spectrum of D lies in $[a, b]$, the spectrum of D_i for each i in $\{1, 2, \cdots, n\}$ lies in $[a, b]$. We have that $\Phi(A) = \Phi(U^* DU) = \text{diag}(\sum_{i=1}^n U_{1i}^* D_i U_{1i}, \cdots, \sum_{i=1}^n U_{ni}^* D_i U_{ni})$. Thus using Lemma 6.2.2, we conclude that the spectrum of $\Phi(A)$ also lies in $[a, b]$ and we can define $f(\Phi(A))$ using the continuous functional calculus.

($\Rightarrow$) Let f be operator-convex. Then by a non-commutative version of Jensen's inequality (in [5]) we have that,

$$f(\sum_{i=1}^{n} U_{ji}^* D_i U_{ji}) \leq \sum_{i=1}^{n} U_{ji}^* f(D_i) U_{ji}, \text{ for all } j \in \{1, 2, \cdots, n\}.$$

Note that $f(\Phi(A)) = \operatorname{diag}(f(\sum_{i=1}^{n} U_{1i}^* D_i U_{1i}), \cdots, f(\sum_{i=1}^{n} U_{ni}^* D_i U_{ni}))$ and $\Phi(f(A)) = \Phi(f(U^*DU)) = \Phi(U^* f(D) U) = \operatorname{diag}(\sum_{i=1}^{n} U_{1i}^* f(D_i) U_{1i}, \cdots, \sum_{i=1}^{n} U_{ni}^* f(D_i) U_{ni})$. From the preceding Jensen's inequality we have that $f(\Phi(A)) \leq \Phi(f(A))$.

($\Leftarrow$) Let $f(\Phi(A)) \leq \Phi(f(A))$ for all self-adjoint elements A in $M_n(\mathscr{R})$ with spectrum in $[a, b]$. Let D_1, D_2 be two self-adjoint elements in $\mathscr{R}$ and $t \in [0, 1]$. We may construct a unitary element V in $M_n(\mathscr{R})$ whose first row is $[\sqrt{t}I \ \ (-\sqrt{1-t})I \ \ 0 \ \cdots \ 0]$ and second row is $[\sqrt{1-t}I \ \ \sqrt{t}I \ \ 0 \ \cdots \ 0]$. Let D be the diagonal element $\operatorname{diag}(D_1, D_2, 0, \cdots, 0)$ in $M_n(\mathscr{R})$ and A be defined by V^*DV. Then as $f(\Phi(A)) \leq \Phi(f(A))$, we have that $f(\Phi(A))_{11} \leq \Phi(f(A))_{11}$ (the $(1, 1)$ entry of corresponding matrices). Thus $f(tD_1 + (1-t)D_2) \leq tf(D_1) + (1-t)f(D_2)$ for any self-adjoint elements D_1, D_2 in $\mathscr{R}$. As $M_m(\mathbb{C}) \hookrightarrow \mathscr{R}$ for any natural number m, f must be m-convex. Thus f is operator convex.

$\square$

Remark 6.2.4. *Note that a II_1 factor $\mathscr{M}$ has copies of $M_m(\mathbb{C})$ for all m in $\mathbb{N}$. We may make this observation by choosing m equivalent orthogonal projections, each with trace $\frac{1}{m}$, and considering the algebra generated by all possible complex linear combinations. The results we have discussed thus hold true in this particular case.*

Theorem 6.2.5. *Let $\mathscr{M}$ be a finite factor with trace τ, and $n \geq 3$ be a natural number. Then a positive contraction A in $\mathscr{M}$ occurs as a diagonal entry of a trace a projection E in $M_n(\mathscr{M})$ if and only if it is in the C^*-polytope generated by $0, I, P_a$ where P_a is a projection in $\mathscr{M}$ such that $\tau(P_a) = \frac{\{na\}}{n}$ ($\{x\} = x - \lfloor x \rfloor$ denotes the fractional part of x).*

Proof. We diagonalize E so that its diagonal form is $D = \mathrm{diag}(I, I, \cdots, I, P_a, 0, \cdots, 0)$ ($= UEU^*$, for some unitary U) where the number of I's is $k := \lfloor na \rfloor$. This is possible as all projections with a given trace are unitarily equivalent in a finite factor ($M_n(\mathscr{M})$ is also a finite factor.)

By Proposition 6.2.1, a diagonal element A, of E is equal to $\sum_{i=1}^n U_1^* U_1 + \cdots + U_k^* U_k + U_{k+1}^* P_a U_{k+1}$ where $U_1, \cdots, U_n$ are operators in $\mathscr{M}$ such that $U_1^* U_1 + \cdots U_n^* U_n = I$. As $U_1^* U_1 + \cdots U_k^* U_k \geq 0$, there is an operator V such that $U_1^* U_1 + \cdots U_k^* U_k = V^* V$. Similarly there is an operator W such that $U_{k+2}^* U_{k+2} + \cdots U_n U_n^* = W^* W$. We may rewrite A as $V^* V + U_k^* P_a U_k + W^* \cdot 0 \cdot W$, $V^* V + U_k^* U_k + W^* W = I$. This proves one direction.

Let $A = V_1^* \cdot I \cdot V_1 + V_2^* P_a V_2 + V_3^* \cdot 0 \cdot V_3$ where $V_1^* V_1 + V_2^* V_2 + V_3^* V_3 = I$. As $n \geq 3$, we may split the term $V_1^* V_1$ into $n-2$ parts *i.e.* $V_1^* V_1 = (\frac{V_1}{\sqrt{n-2}})^* \frac{V_1}{\sqrt{n-2}} + \cdots + (\frac{V_1}{\sqrt{n-2}})^* \frac{V_1}{\sqrt{n-2}}$ ($n-2$ times). By Theorem 3.3.3, there is a unitary operator V in $M_n(\mathscr{M})$ with first row as $[(\frac{V_1}{\sqrt{n-2}})^* \ \cdots \ (\frac{V_1}{\sqrt{n-2}})^* \ \ V_2^* \ \ V_3^*]$. The operator A occurs in the diagonal of VDV^* which is a trace a projection.

□

Remark 6.2.6. *When $n = 2$, we have explicitly found the diagonal blocks of projections in the case of $\mathscr{M} = M_k(\mathbb{C})$. In fact, we may conclude from Theorem 6.1.6 and the proof of the above theorem that although a C^*-segment joining two projections need not be convex, but the set of spectral distributions of elements in the C^*-segment is a convex set.*

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